CHAPTER 2

Periodic and transient signals

2.1 Introduction

Signals and waveforms are central to communications. A *signal* is defined [Hanks] as 'any sign, gesture, token, etc., that serves to communicate information'. It will be shown later that to communicate information such symbols must be in some sense unpredictable or random. The word signal, as applied to electronic communications, therefore implies an electrical quantity (e.g. voltage) possessing some characteristic (e.g. amplitude) which varies unpredictably. A *waveform* is defined as 'the shape of a wave or oscillation obtained by plotting the value of some changing quantity against time'. In electronic communications the term waveform implies an electrical quantity which varies *periodically*, and therefore predictably. Strictly this precludes a waveform from conveying information. However, a waveform can be adapted to convey information by varying one or more of its parameters in sympathy with a signal. Such waveforms are called carriers and typically consist of a sinusoid or pulse train modulated in amplitude, phase or frequency.

Fluctuating voltages and currents can be alternatively classified as either periodic or aperiodic. A periodic signal, if shifted by an appropriate time interval, is unchanged. An aperiodic signal does not possess this property. In this context the term periodic signal is clearly synonymous with waveform. In this chapter our principal concern is with periodic signals and one type of aperiodic signal, i.e. transients. A transient signal is one which has a well defined location in time. This does not necessarily mean it must be zero outside a certain time interval but it does imply that the signal at least tends to zero as time tends to $\pm\infty$. The one sided decaying exponential function is an example of a transient signal which has a well defined start and tends to zero as $t \rightarrow \infty$.

If a signal's parameters (amplitude, shape and phase in the case of a periodic signal, amplitude, shape and location in the case of a transient signal) are known, then the signal is said to be deterministic. This means that, in the absence of noise, any future value of the signal can be determined precisely. Signals which are not deterministic must be described using probability theory, as discussed in Chapter 3.

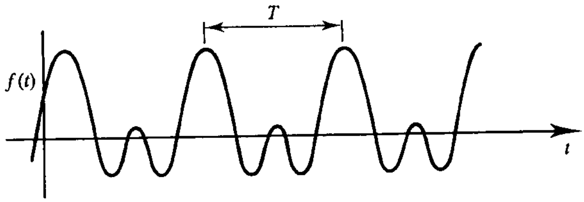


Figure 2.1 *Example of a periodic signal.*

2.2 Periodic signals

A periodic signal is defined as one which has the property:

$$f(t) = f(t \pm nT) \tag{2.1}$$

where n is any integer and T is the repetition period (or simply period) of the signal, Figure 2.1. A consequence of this definition is that periodic signals have no starting time or finishing time, i.e. they are eternal. The normalised power, P, averaged over any T second period, is:

$$P = \frac{1}{T} \int_{t}^{t+T} |f(t)|^2 \, dt \quad (\mathrm{V}^2) \tag{2.2}$$

where the integral is the normalised energy per period. This is clearly a well defined finite quantity. The total energy, E, in a periodic signal, however, is infinite, i.e.:

$$E = \int_{-\infty}^{\infty} |f(t)|^2 \, dt = \infty \quad (\mathrm{V}^2\ \mathrm{s}) \tag{2.3}$$

For this reason periodic signals (along with some other types of signal) are sometimes called *power* signals. It also means that signals which are strictly periodic are unrealisable. The concept of a strictly periodic signal is, however, both simple and useful. Furthermore it is easy to generate signals which approximate very closely the conceptual ideal.

2.2.1 Sinusoids, cisoids and phasors

An especially simple and useful set of periodic signals is the set of sinusoids. These are generated naturally by projecting a point P, located on the circumference of a rotating disc (with unit radius), onto various planes, Figure 2.2.

If the length of OA in Figure 2.2 is plotted against angular position θ, then the result is the function cos θ, Figure 2.3(a). If the length of OB is plotted against θ, then the result is sin θ, Figure 2.3(b). (If the length of $O'C$ on the plane tangent to the disc is plotted against θ then the function tan θ results.) If the disc is not of unit radius then the normal (circular) trigonometric ratios are defined by:

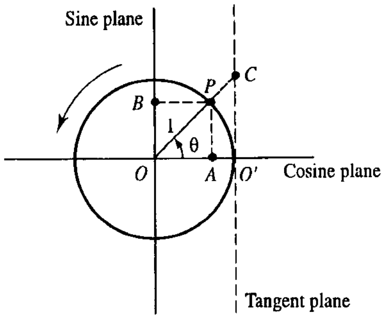


Figure 2.2 *Generation of sinusoids by projection of a radius onto perpendicular planes.*

$$\cos\theta = \frac{OA}{OO'} \tag{2.4(a)}$$

$$\sin\theta = \frac{OB}{OO'} \tag{2.4(b)}$$

The angle θ, expressed in degrees or radians, is called the phase of the function and can be related to the time period, T, taken for one revolution, i.e.:

$$\theta = 360\,\frac{t}{T} \text{ degrees} \tag{2.5(a)}$$

$$\theta = 2\pi\,\frac{t}{T} \text{ radians} \tag{2.5(b)}$$

The angular velocity (or radian frequency), $\omega = d\theta/dt$, of the disc is therefore given by:

$$\omega = \frac{2\pi}{T} \text{ rad/s} \tag{2.6}$$

and angular position or phase by:

$$\theta = \omega t \quad \text{rad} \tag{2.7}$$

$1/T$ is the cyclical frequency of the disc in cycles/s or Hz. The sine and cosine functions plotted against time, t, are shown in Figure 2.4. The functions cos θ and sin θ are identical in shape but cos θ reaches its peak value $T/4$ seconds (i.e. $\pi/2$ radians or 90°) *before* sin θ. cos θ is therefore said to *lead* sin θ by $\pi/2$ radians and, conversely, sin θ is said to *lag* cos θ by $\pi/2$ radians. The relationship between cosine and sine functions can be summarised by:

$$\cos\theta = \sin(\theta + \pi/2) \tag{2.8}$$

Notice that the cosine function and sine function have even and odd symmetry respectively about $t = 0$, i.e.:

$$\cos\theta = \cos(-\theta) \tag{2.9(a)}$$

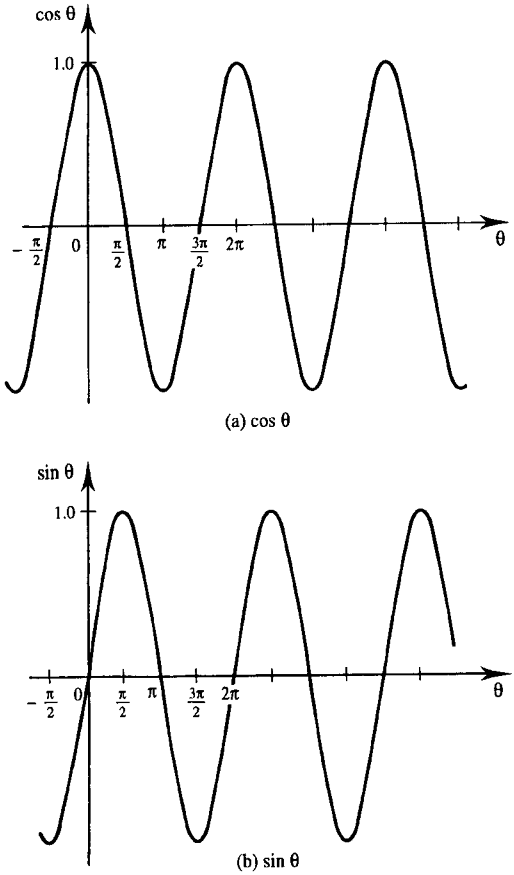


Figure 2.3 *Circular trigonometric functions plotted against phase: (a) cosine function of phase angle; (b) sine function of phase angle.*

$$\sin \theta = -\sin(-\theta) \tag{2.9(b)}$$

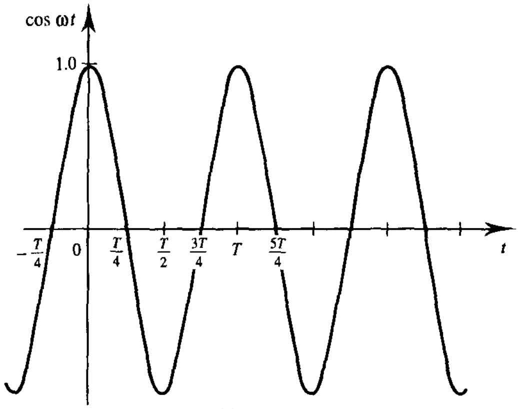
A cisoid is a general term which describes a rotating vector in the complex plane. Figure 2.5 shows a cisoid (which makes an angle ϕ with the plane's real axis at time $t = 0$) resolved onto real and imaginary axes. From the definition of the circular trigonometric functions it is clear that the component resolved onto the real axis is:

$$\Re[\text{cisoid}] = \cos(\omega t + \phi) \tag{2.10(a)}$$

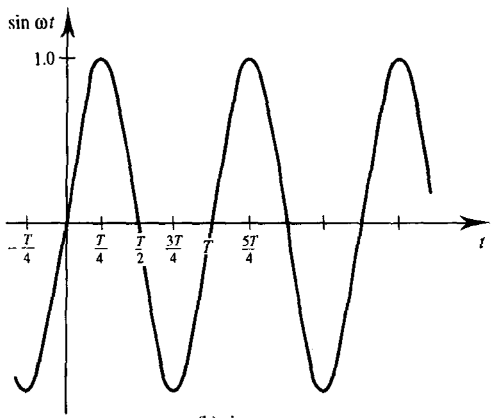
and the component resolved onto the imaginary axis is:

$$\Im[\text{cisoid}] = \sin(\omega t + \phi) \tag{2.10(b)}$$

Using Euler's formula (which relates geometrical and algebraic quantities) the real and imaginary components can be expressed together as:



(a) cos ωt



(b) sin ωt

Figure 2.4 *Circular trigonometric functions plotted against time: (a) cosine function of time; (b) sine function of time.*

$$\cos(\omega t + \phi) + j\,\sin(\omega t + \phi) = e^{j(\omega t + \phi)} \tag{2.11}$$

Equation (2.11) is the origin of the term cisoid which is a contraction of (*c*os + *i s*in)us*oid*, where $i = \sqrt{-1}$ replaces j. In three dimensions, with time (or phase) progressing along the axis perpendicular to the complex plane, the cisoid traces out a helical curve, Figure 2.6. For $\phi = 0$ the projection of this helix onto the imaginary/time plane is a sine wave and its projection onto the real/time plane is a cosine wave.

There is a satisfying symmetry relating real sinusoids and complex cisoids in that two, quadrature, sinusoids are required to generate a single cisoid and two, counter rotating, cisoids are required to generate a single sinusoid. If the cisoids are a conjugate pair then the resulting sinusoid is purely real, Figure 2.7.

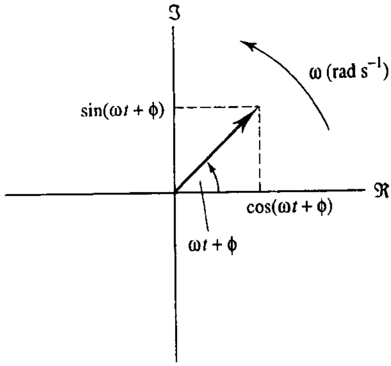


Figure 2.5 *Rotating vector or cisoid.*

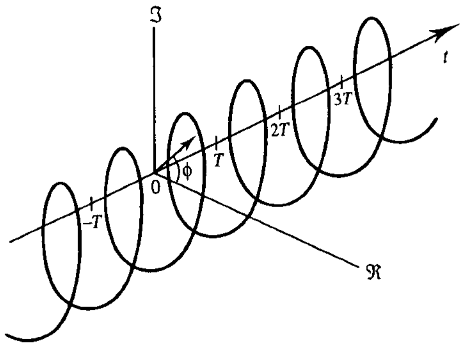


Figure 2.6 *Sketch of cisoid with time progressing perpenducular to the complex plane.*

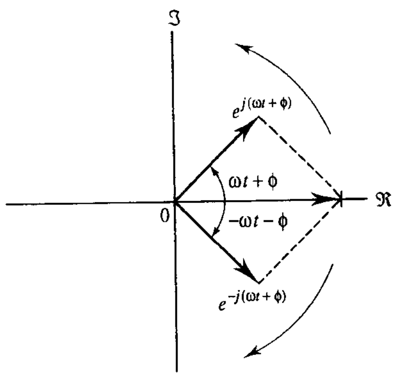


Figure 2.7 *Synthesis of real sinusoid wave from two counter-rotating, conjugate, cisoids.*

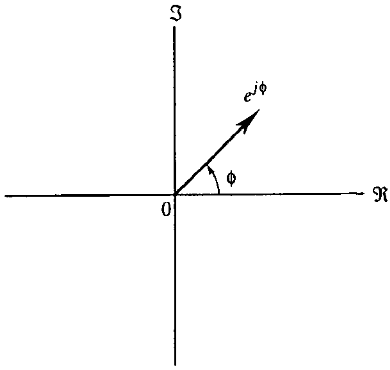


Figure 2.8 *Phasor corresponding to* $e^{j(\omega t+\phi)}$.

Phasors are cisoids which have had their time dependence suppressed. The phasor corresponding to $e^{j(\omega t+\phi)}$ is therefore $e^{j\phi}$ and corresponds to an instantaneous picture of the cisoid at the time $t = 0$, Figure 2.8. Another interpretation of phasors is that they represent a cisoid drawn in a plane which is itself rotating at the same angular frequency as the cisoid. The phasor is therefore stationary with respect to the complex plane. The close relationship between cisoids and phasors is such that a distinction between them is rarely made in practice, the term phasor often being used to describe both.

2.2.2 Fourier series

Almost any periodic signal of practical interest can be approximated by adding together sinusoids with the correct frequencies, amplitudes and phases. An example of a saw-tooth waveform approximated by a sum of sinusoids is shown in Figure 2.9. In general the error between the synthesised approximation and the actual waveform can be made as small as desired by including enough sinusoids in the sum. (This is not true at points of discontinuity, however: see section 2.2.4.) Only one sinusoid at each integer multiple of the fundamental frequency is required in the sum, providing that its amplitude and phase can be chosen freely. The fundamental frequency, f_1, is the reciprocal of the waveform's period, T, i.e.:

$$f_1 = 1/T \tag{2.12}$$

The sinusoid with frequency $f_n = nf_1$ is called the nth harmonic of the fundamental. If the waveform being approximated has a non-zero mean value then, in addition to the set of sinusoids, a 0 Hz, constant, or DC term must be included in the sum. In general, then, the sinusoidal sum, which is called a Fourier series, is given by:

$$v(t) = C_0 + C_1 \cos(\omega_1 t + \phi_1) + C_2 \cos(\omega_2 t + \phi_2) + \cdots \tag{2.13}$$

where C_0 (V) is the DC term, $\omega_1 = 2\pi/T$ (rad/s) is the fundamental frequency and $\omega_2 = 2(2\pi/T)$ (rad/s) is the second harmonic frequency, etc. The series may be truncated

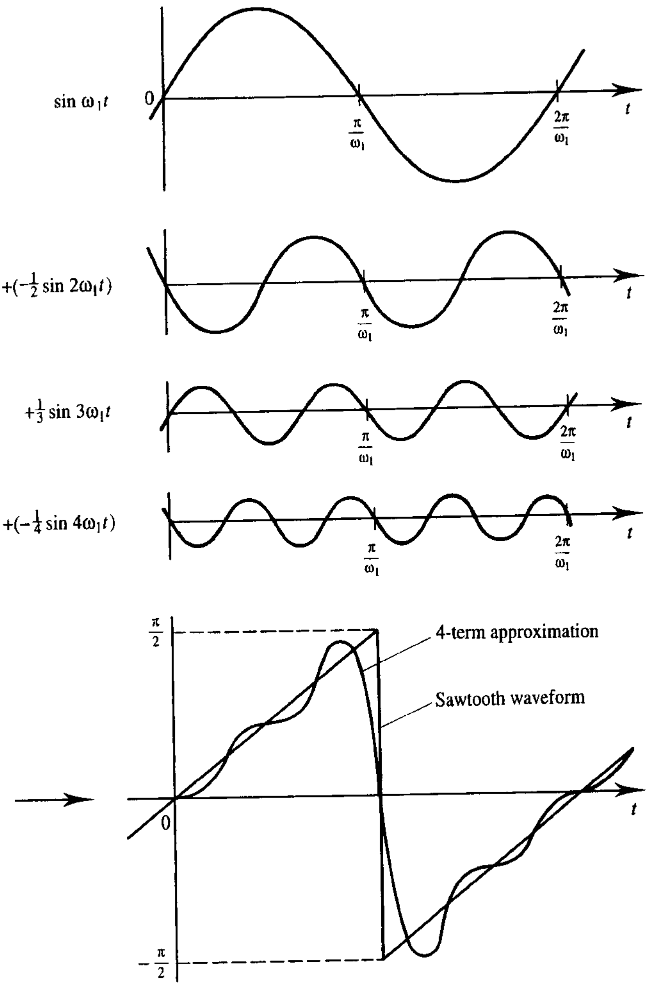


Figure 2.9 *Synthesis of sawtooth waveform by addition of harmonically related sinusoids.*

after a finite number of terms or may extend indefinitely.

Trigonometric forms

The trigonometric form of the Fourier series, expressed by equation (2.13), can be written more compactly as:

$$v(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(\omega_n t + \phi_n) \qquad (2.14(a))$$

This is the *cosine* form of the series since each term is written as a cosine function (with an explicit phase angle, ϕ). Since each term in the periodic signal is a harmonic of the fundamental, equation (2.14(a)) can be rewritten as:

$$v(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_1 t + \phi_n) \qquad (2.14(b))$$

A slightly different trigonometric form can be created by resolving each sinusoid into cosine and sine components (each with zero phase angle). This gives the *cosine – sine* form of the trigonometric Fourier series, i.e.:

$$v(t) = C_0 + \sum_{n=1}^{\infty} (A_n \cos \omega_n t - B_n \sin \omega_n t) \qquad (2.14(c))$$

Notice that the series is still specified by two real numbers per harmonic but in this case the numbers are cosine and sine amplitudes (or inphase and quadrature amplitudes) rather than amplitude and phase. (The use of a minus sign in equation (2.14(c)) may seem eccentric but its advantage will become clear later.)

If the amplitude, C_n, of the cosine Fourier series is plotted against frequency, $f_n = \omega_n/2\pi$ (Hz), the result is called a discrete, or line, amplitude spectrum, Figure 2.10(a). Similarly, if ϕ_n is plotted against f_n the result is a discrete phase spectrum, Figure 2.10(b). Notice that, for obvious reasons, the phase of the DC (0 Hz) component is not defined. Notice also that the height of the lines in the amplitude spectrum of Figure 2.10(a) represents the peak values of the sinusoidal components. It is possible, of course, to define an RMS amplitude spectrum which would be the same as the peak amplitude spectrum except that each line would be smaller by a factor of $1/\sqrt{2}$.

If the sinusoids of a cosine series are displayed in three dimensions, plotted against time and frequency, Figure 2.11, then the amplitude spectrum corresponds to a projection onto the amplitude-frequency plane. This gives a picture of the 'frequency content' of a signal.

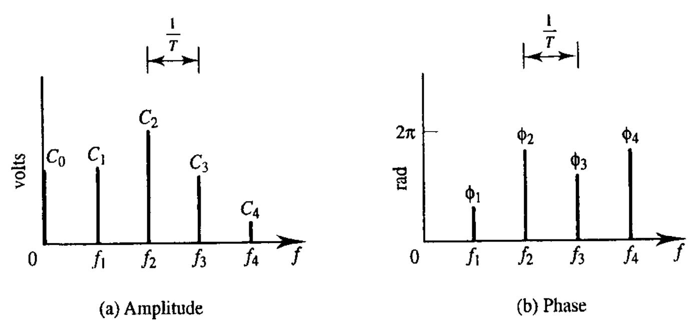


Figure 2.10 *Discrete spectrum of a periodic signal.*

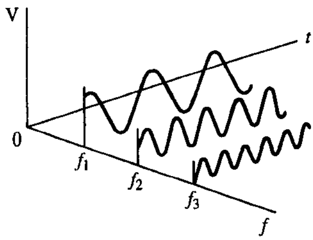


Figure 2.11 *Component sinusoids of a Fourier series plotted against time and frequency.*

Calculation of Fourier coefficients

Since C_0 is the DC, or average, value of the waveform being approximated it is clear that it can be calculated using:

$$C_0 = \frac{1}{T} \int_{t}^{t+T} v(t)\, dt \tag{2.15}$$

In practice it is easier to calculate the A_n and B_n coefficients associated with the cosine-sine form of the Fourier series than to find the C_n and ϕ_n values of the cosine form. (C_n and ϕ_n can be easily calculated from A_n and B_n as will be shown later.) The essential task in calculating the value of A_1, for example, is to find out how much of the inphase fundamental component, $\cos \omega_1 t$, is contained in $v(t)$. In other words the similarity between $\cos \omega_1 t$ and $v(t)$ must be established. One way of quantifying this similarity is to find their mean product, i.e. $\langle v(t) \cos \omega_1 t \rangle$ where $\langle\ \rangle$ signifies a time average. If $v(t)$ tends to be positive when $\cos \omega_1 t$ is positive and negative when $\cos \omega_1 t$ is negative then $\langle v(t) \cos \omega_1 t \rangle$ will tend to be large and positive indicating a large degree of similarity, Figure 2.12. This would suggest that $v(t)$ contained a large $\cos \omega_1 t$ component. If, conversely, $v(t)$ tends to be negative when $\cos \omega_1 t$ is positive and vice versa then $\langle v(t) \cos \omega_1 t \rangle$ will tend to be large and negative. This would indicate extreme dissimilarity and the conclusion would be that $v(t)$ contained a large $-\cos \omega_1 t$

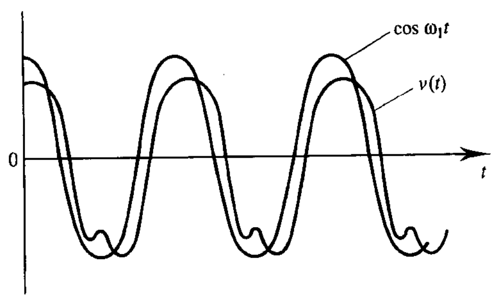


Figure 2.12 *Similar waveforms: $v(t)$ and cos $\omega_1 t$.*

component. If there was little correlation between the polarity of $v(t)$ and $\cos \omega_1 t$ then $\langle v(t) \cos \omega_1 t \rangle$ would be close to zero and the conclusion would be that $v(t)$ contained almost no $\cos \omega_1 t$ component.

The normal way to find an average value is given by equation (2.15). Therefore:

$$\langle v(t) \cos \omega_1 t \rangle = \frac{1}{T} \int_t^{t+T} v(t) \cos \omega_1 t \, dt \tag{2.16}$$

To find the Fourier coefficient, A_1, however, the actual equation used is:

$$A_1 = \frac{2}{T} \int_t^{t+T} v(t) \cos \omega_1 t \, dt \tag{2.17}$$

This is because if $v(t)$ was *exactly* like $\cos \omega_1 t$ (i.e. $v(t) = \cos \omega_1 t$) then A_1 should be 1.0. Unfortunately:

$$\langle \cos^2 \omega_1 t \rangle = \langle ½(1 + \cos 2\omega_1 t) \rangle = ½ \tag{2.18}$$

The factor of two in equation (2.17) is necessary to make $A_1 = 1$. The general formulae for calculating the cosine-sine Fourier coefficients are therefore:

$$A_n = \frac{2}{T} \int_t^{t+T} v(t) \cos \omega_n t \, dt \tag{2.19(a)}$$

$$B_n = -\frac{2}{T} \int_t^{t+T} v(t) \sin \omega_n t \, dt \tag{2.19(b)}$$

(B_n quantifies the similarity between $v(t)$ and $-\sin \omega_n t$.) If the cosine series is required the values of C_n and ϕ_n are found easily using simple trigonometry, Figure 2.13, i.e.:

$$C_n = \sqrt{(A_n^2 + B_n^2)} \tag{2.20(a)}$$

$$\phi_n = \tan^{-1}(B_n/A_n) \tag{2.20(b)}$$

A satisfying engineering interpretation of equations (2.19(a)) and (b) is that of 'filtering integrals'. If $v(t)$ is made up of many harmonically related sinusoids the average product of each of these sinusoids with $\cos \omega_n t$ or $\sin \omega_n t$ is only non-zero for that sinusoid which has the same frequency as $\cos \omega_n t$ or $\sin \omega_n t$. This *orthogonality* property is summarised mathematically by:

$$\frac{2}{T} \int_t^{t+T} \cos \omega_m t \cos \omega_n t \, dt = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases} \tag{2.21(a)}$$

$$\frac{2}{T} \int_t^{t+T} \sin \omega_m t \sin \omega_n t \, dt = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases} \tag{2.21(b)}$$

$$\frac{2}{T} \int_t^{t+T} \cos \omega_m t \sin \omega_n t \, dt = 0 \tag{2.21(c)}$$

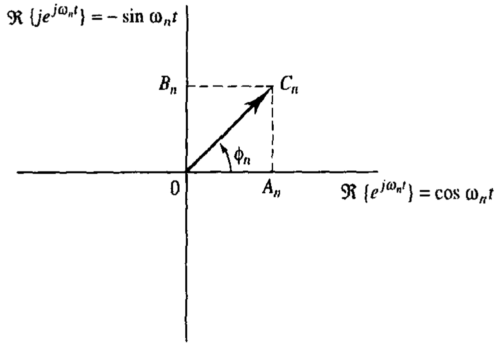


Figure 2.13 *Relationship between amplitude (C_n) of Fourier coefficients and inphase and quadrature components (A_n and B_n).*

These properties and their geometrical interpretation will be discussed further, in a more general context, in section 2.5.

EXAMPLE 2.1

Find the first two Fourier coefficients of a unipolar rectangular pulse train with amplitude 3 V, period 10 ms, duty cycle 20% and pulse leading edge at time $t = 0$. The pulse train $v(t)$ is shown in Figure 2.14.

The DC term is given by:

$$
\begin{aligned}
C_0 &= \frac{1}{T}\int_t^{t+T} v(t)\,dt \\
&= \frac{1}{0.01}\int_0^{0.002} 3\,dt \\
&= 100\,[3\,t]_0^{0.002} = 0.6 \ \ (\mathrm{V})
\end{aligned}
$$

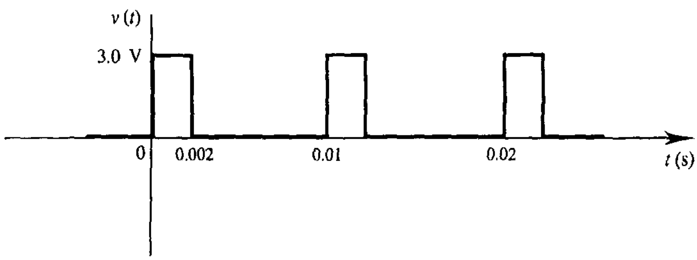


Figure 2.14 *Periodic rectangular pulse train for Example 2.1.*

The inphase coefficients are given by:

$$A_1 = \frac{2}{T}\int_t^{t+T} v(t)\cos 2\pi f_1 t\; dt$$

$$= \frac{2}{0.01}\int_0^{0.002} 3\cos\left(2\pi\frac{1}{0.01}t\right)dt$$

$$= 600\left[\frac{\sin(2\pi\,100\,t)}{2\pi\,100}\right]_0^{0.002} = 0.9082 \quad (\text{V})$$

$$A_2 = \frac{2}{T}\int_t^{t+T} v(t)\cos 2\pi f_2 t\; dt$$

$$= \frac{2}{0.01}\int_0^{0.002} 3\cos\left(2\pi\frac{2}{0.01}t\right)dt$$

$$= 600\left[\frac{\sin(2\pi\,200\,t)}{2\pi\,200}\right]_0^{0.002} = 0.2806 \quad (\text{V})$$

The quadrature coefficients are given by:

$$B_1 = -\frac{2}{T}\int_t^{t+T} v(t)\sin 2\pi f_1 t\; dt$$

$$= -\frac{2}{0.01}\int_0^{0.002} 3\sin\left(2\pi\frac{1}{0.01}t\right)dt$$

$$= -600\left[\frac{-\cos(2\pi\,100\,t)}{2\pi\,100}\right]_0^{0.002} = \frac{3}{\pi}[0.3090 - 1] = -0.6599 \quad (\text{V})$$

$$B_2 = -\frac{2}{T}\int_t^{t+T} v(t)\sin 2\pi f_2 t\; dt$$

$$= -\frac{2}{0.01}\int_0^{0.002} 3\sin\left(2\pi\frac{2}{0.01}t\right)dt$$

$$= -600\left[\frac{-\cos(2\pi\,200\,t)}{2\pi\,200}\right]_0^{0.002} = \frac{3}{2\pi}[-0.8090 - 1] = -0.8637 \quad (\text{V})$$

The Fourier coefficient amplitudes are given in equation (2.20(a)) by:

$$C_n = \sqrt{(A_n^2 + B_n^2)}$$

i.e.:

$$C_0 = 0.6 \text{ V}$$

$$C_1 = \sqrt{(0.9082^2 + 0.6599^2)} = 1.1226 \text{ V}$$

$$C_2 = \sqrt{(0.2806^2 + 0.8637^2)} = 0.9081 \text{ V}$$

and the Fourier coefficient phases are given in equation (2.20(b)) by:

$$\phi_n = \tan^{-1}\left(\frac{B_n}{A_n}\right)$$

i.e.:

$$\phi_1 = \tan^{-1}\left(\frac{-0.6599}{0.9082}\right) = -0.6284 \text{ rad or } -36.0°$$

$$\phi_2 = \tan^{-1}\left(\frac{-0.8637}{0.2806}\right) = -1.257 \text{ rad or } -72.0°.$$

Note that moving the pulse train to the right or left will change the phase spectrum but not the amplitude spectrum. For example, if the pulse train is moved 0.001 s to the left (such that it has even symmetry about $t = 0$) then the Fourier series will contain cosine waves only and the phase spectrum will be restricted to values of 0° and 180°.

Figure 2.9 shows the decomposition of a sawtooth wave into terms up to the fourth harmonic and also includes the wave reconstructed from these components.

Exponential form

As an alternative to calculating the A_n and B_n coefficients of the cosine-sine Fourier series separately they can be calculated together using:

$$A_n + jB_n = \frac{2}{T} \int_t^{t+T} v(t)(\cos \omega_n t - j \sin \omega_n t) \, dt \tag{2.22}$$

This corresponds to synthesising the function $v(t)$ from the real part of a set of harmonically related cisoids, i.e.:

$$v(t) = C_0 + \sum_{n=1}^{\infty} (A_n \cos \omega_n t - B_n \sin \omega_n t)$$

$$= C_0 + \Re\left\{ \sum_{n=1}^{\infty} \tilde{C}_n \, e^{j\omega_n t} \right\} \tag{2.23}$$

where $\tilde{C}_n = A_n + jB_n$. The tilde (˜) indicates that $\tilde{C}_n$ is generally complex.

Having a separate DC term, C_0, in equation (2.23) and being required to take the real part of the other terms is, at best, a little inelegant. This can be overcome, however, by using a pair of counter-rotating, conjugate cisoids to represent each real sinusoid in the series, i.e.:

$$v(t) = \sum_{n=-\infty}^{\infty} \tilde{C}'_n \, e^{j\omega_n t} \tag{2.24(a)}$$

where:

$$\tilde{C}'_n = \begin{cases} \tilde{C}_n/2 & \text{for } n > 0 \\ C_0 & \text{for } n = 0 \\ \tilde{C}^*_{-n}/2 & \text{for } n < 0 \end{cases} \tag{2.24(b)}$$

Thus, for example, the pair of cisoids corresponding to $n = \pm 3$ (i.e. the third harmonic cisoids) may look like those shown in Figure 2.15. Notice that the magnitude, $|\tilde{C}'_n|$, of each cisoid in the formulation of equations (2.24) is half that, $|\tilde{C}_n|$, of the corresponding cisoids in equation (2.23) or the corresponding sinusoids in equations (2.14(a)) and (b). Thus the formula for the calculation of Fourier (exponential) coefficients gives results only half as large as that for the trigonometric series, i.e.:

$$\tilde{C}'_n = \frac{1}{T} \int_t^{t+T} v(t) \, e^{-j\omega_n t} \, dt \tag{2.25}$$

$e^{-j\omega_n t}$, here, filters out that part of $v(t)$ which is identical to $e^{j\omega_n t}$ (since $(1/T)\int_0^T e^{j\omega_n t} e^{-j\omega_n t} \, dt = 1$). When n is positive, the positively rotating (i.e. anticlockwise) cisoids are obtained and when n is negative (remembering that $\omega_n = n\omega_1$) the negatively rotating (i.e. clockwise) cisoids are obtained. When $n = 0$ equation (2.25) gives the DC term.

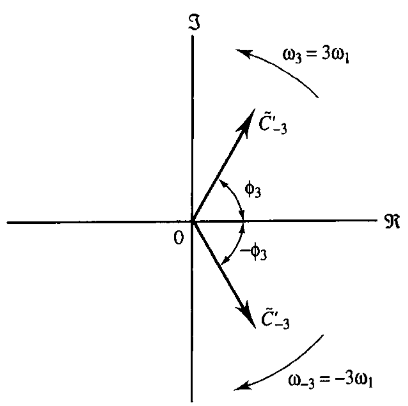


Figure 2.15 *Pair of counter-rotating, conjugate, cisoids (drawn for t = 0) corresponding to the (real) third harmonic of a periodic signal.*

Double sided spectra

If the amplitudes $|\tilde{C}'_n|$ of the cisoids found using equation (2.25) are plotted against frequency f_n the result is called a double (or two) sided (voltage) spectrum. Such a spectrum is shown in Figure 2.16. If $v(t)$ is purely real, i.e. it represents a signal that can exist in a practical single channel system, then each positively rotating cisoid is matched by a conjugate, negatively rotating, cisoid which cancels the imaginary part to zero. The double sided amplitude spectrum thus has *even* symmetry about 0 Hz, Figure 2.16(a). The single sided spectrum (positive frequencies only) representing the amplitudes of the real sinusoids in the trigonometric Fourier series can be found from the double sided spectrum by folding over the negative frequencies of the latter and adding them to the positive frequencies.

If the phase angles of the cisoids are plotted against f_n the result is the double sided phase spectrum which, due to the conjugate pairing of cisoids, will have *odd* symmetry for real waveforms, Figure 2.16(b).

The even and odd symmetries of the amplitude and phase spectra of real signals are summarised by:

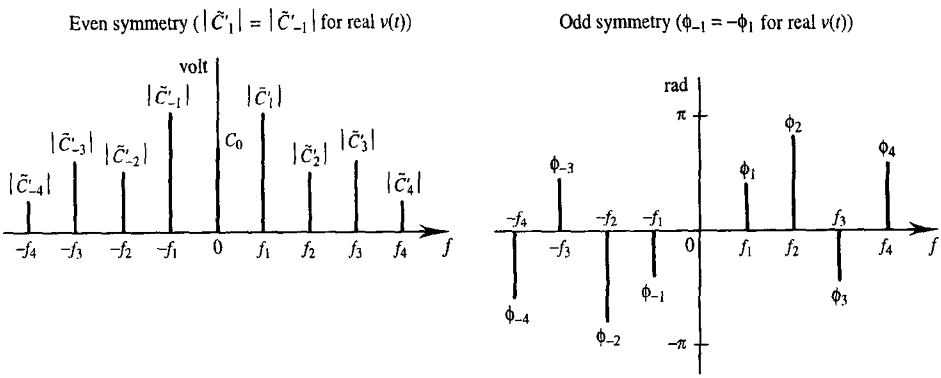
$$|\tilde{C}'_n| = |\tilde{C}'_{-n}| \tag{2.26(a)}$$

and:

$$\arg(\tilde{C}'_n) = -\arg(\tilde{C}'_{-n}) \tag{2.26(b)}$$

Calculation of coefficients for waveforms with symmetry

For a waveform $v(t)$ with certain symmetry properties, the calculation of some, or all, of the Fourier coefficients is simplified. These symmetries and the corresponding simplifications for the calculation of C_0, A_n and B_n are shown in Table 2.1.

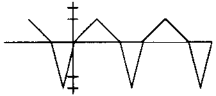
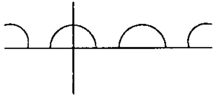
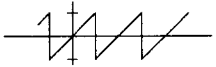
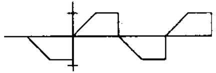


(a) Double sided amplitude spectrum

(b) Double sided phase spectrum

Figure 2.16 *Double sided voltage spectrum of a real periodic signal.*

Table 2.1 *Fourier series formulae for waveforms with symmetry*

Type of Symmetry	Definition	Example $f(t)$	C_o	A_n	B_n	Non-zero terms
Zero mean	$\int_0^T f(t)\,dt = 0$		$C_o = 0$	$A_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cos\omega_n t\,dt$	$B_n = -\frac{T}{2}\int_{\frac{T}{2}}^{\frac{T}{2}} f(t)\sin\omega_n t\,dt$	A_n and B_n
Even	$f(t) = f(-t)$		$C_o = \frac{1}{T}\int_0^T f(t)\,dt$	$A_n = \frac{4}{T}\int_0^{\frac{T}{2}} f(t)\cos\omega_n t\,dt$	$B_n = 0$	A_n and C_o
Odd	$f(t) = -f(-t)$		$C_o = 0$	$A_n = 0$	$B_n = -\frac{4}{T}\int_0^{\frac{T}{2}} f(t)\sin\omega_n t\,dt$	B_n
Half-wave	$f(t) = -f\left(t + \frac{T}{2}\right)$		$C_o = 0$	$A_n = \frac{4}{T}\int_0^{\frac{T}{2}} f(t)\cos\omega_n t\,dt$	$B_n = -\frac{4}{T}\int_0^{\frac{T}{2}} f(t)\sin\omega_n t\,dt$	A_n and B_n, odd harmonics (n odd) only

Discrete power spectra and Parseval's theorem

If the trigonometric coefficients are divided by $\sqrt{2}$ and plotted against frequency the result is a one sided, RMS amplitude spectrum. If each RMS amplitude is then squared, the height of each spectral line will represent the normalised power (or power dissipated in 1 Ω) associated with that frequency, i.e.:

$$\left\{ \frac{|\tilde{C}_n|}{\sqrt{2}} \right\}^2 = \frac{|\tilde{C}_n|^2}{2} = P_{n1} \ (\mathrm{V}^2) \tag{2.27}$$

Such a one sided power spectrum is illustrated in Figure 2.17(a). A double sided version of the power spectrum can be defined by associating half the power in each line, $P_{n2} = |\tilde{C}_n|^2/4$, with negative frequencies, Figure 2.17(b). The double sided power spectrum is therefore obtained from the double sided amplitude spectrum by squaring each cisoid amplitude, i.e. $P_{n2} = |\tilde{C}'_n|^2$. That the total power in an entire line spectrum is the sum of the powers in each individual line might seem an intuitively obvious statement. This is true, however, only because of the orthogonal nature of the individual sinusoids making up a periodic waveform. Obvious or not, this statement, which applies to any periodic signal, is known as Parseval's theorem. It can be stated in several forms, one of the most useful being:

$$\text{Total power, } P = \frac{1}{T} \int_t^{t+T} |v(t)|^2 \, dt = \sum_{n=-\infty}^{\infty} |\tilde{C}'_n|^2 \tag{2.28}$$

The proof of Parseval's theorem is straightforward and is given below:

$$P = \frac{1}{T} \int_t^{t+T} v(t) \, v^*(t) \, dt \tag{2.29(a)}$$

$$v^*(t) = \left[\sum_{n=-\infty}^{\infty} \tilde{C}'_n \, e^{j\omega_n t} \right]^*$$

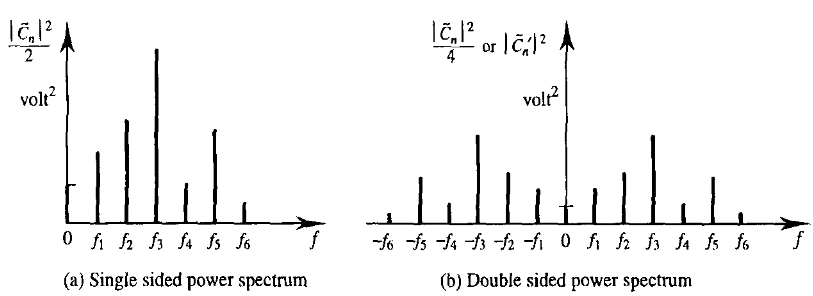


Figure 2.17 *Power spectra of a periodic signal.*

$$= \sum_{n=-\infty}^{\infty} (\tilde{C}_n'^{*} \; e^{-j\omega_n t}) \tag{2.29(b)}$$

Therefore:

$$P = \frac{1}{T} \int_{t}^{t+T} v(t) \sum_{n=-\infty}^{\infty} (\tilde{C}_n'^{*} \; e^{-j\omega_n t}) \; dt$$

$$= \sum_{n=-\infty}^{\infty} \left[\frac{1}{T} \int_{t}^{t+T} v(t) \, e^{-j\omega_n t} \; dt \;\; \tilde{C}_n'^{*} \right]$$

$$= \sum_{n=-\infty}^{\infty} \tilde{C}_n' \; \tilde{C}_n'^{*} \tag{2.29(c)}$$

EXAMPLE 2.2

Find the 0 Hz and first two harmonic terms in the double sided spectrum of the half-wave rectified sinusoid shown in Figure 2.18(a) and sketch the resulting amplitude and phase spectra. What is the total power in these components?

$$\tilde{C}_0' = \frac{1}{T} \int_{t}^{t+T} v(t) \, e^{-j2\pi\, 0 t} \; dt$$

$$= \frac{1}{0.02} \int_{-0.0025}^{0.0075} \sin\left(2\pi \, \frac{1}{0.02} \, t + 2\pi \, \frac{0.0025}{0.02} \right) dt$$

$$= 50 \int_{-0.0025}^{0.0075} \sin(100\pi t + 0.25\pi) \; dt$$

$$= 50 \int_{-0.0025}^{0.0075} \sin(100\pi t) \cos(0.25\pi) + \cos(100\pi t) \sin(0.25\pi) \; dt$$

$$= 50 \, \frac{1}{\sqrt{2}} \left[\frac{-\cos 100\pi t}{100\pi} \right]_{-0.0025}^{0.0075} + 50 \, \frac{1}{\sqrt{2}} \left[\frac{\sin 100\pi t}{100\pi} \right]_{-0.0025}^{0.0075}$$

$$= \frac{50}{100\pi} \, \frac{1}{\sqrt{2}} \{ -[-0.7071 - 0.7071] + [0.7071 - (-0.7071)] \}$$

$$= \frac{50}{100\pi} \, \frac{1}{\sqrt{2}} \, 4 \, (0.7071) \; = \; \frac{1}{\pi} \; (= 0.3183) \;\; (\mathrm{V})$$

$$\tilde{C}_1' = \frac{1}{T} \int_{t}^{t+T} v(t) \, e^{-j2\pi f_1 t} \; dt$$

$$= \frac{1}{0.02} \int_{-0.0025}^{0.0075} \sin\left(2\pi \, \frac{1}{0.02} \, t + \frac{2\pi \; 0.0025}{0.02} \right) e^{-j2\pi \frac{1}{0.02} t} \; dt$$

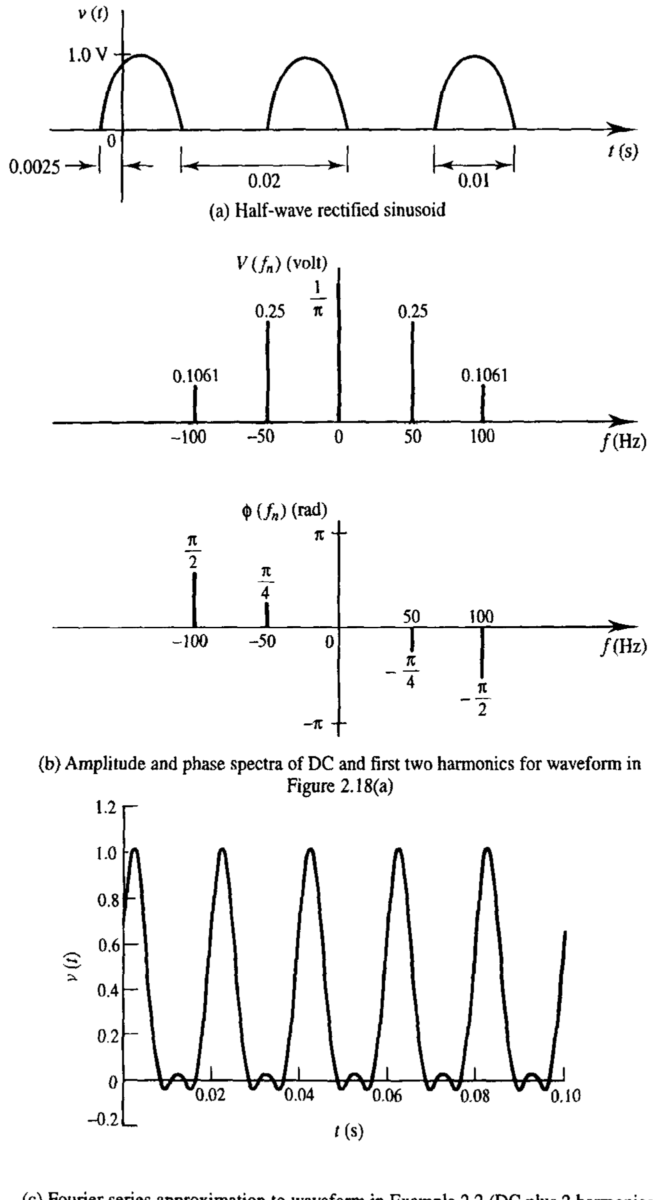


Figure 2.18 *Waveform, spectra and Fourier series approximation for Example 2.2.*

$$= \frac{50}{\sqrt{2}} \int_{-0.0025}^{0.0075} \sin(100\pi t)\,(\cos 100\pi t - j \sin 100\pi t)\, dt$$

$$+ \frac{50}{\sqrt{2}} \int_{-0.0025}^{0.0075} \cos(100\pi t)\,(\cos 100\pi t - j \sin 100\pi t)\, dt$$

$$= \frac{25}{\sqrt{2}} \left[\int_{-0.0025}^{0.0075} \sin(200\pi t)\, dt + \int_{-0.0025}^{0.0075} 1 + \cos(200\pi t)\, dt \right]$$

$$- j\, \frac{25}{\sqrt{2}} \left[\int_{-0.0025}^{0.0075} (1 - \cos(200\pi t))\, dt + \int_{-0.0025}^{0.0075} \sin(200\pi t)\, dt \right]$$

$$= \frac{25}{\sqrt{2}} \left\{ \left[\frac{-\cos(200\pi t)}{200\pi} \right]_{-0.0025}^{0.0075} + [\, t\,]_{-0.0025}^{0.0075} + \left[\frac{\sin(200\pi t)}{200\pi} \right]_{-0.0025}^{0.0075} \right.$$

$$\left. - j[\, t\,]_{-0.0025}^{0.0075} + j \left[\frac{\sin(200\pi t)}{200\pi} \right]_{-0.0025}^{0.0075} - j \left[\frac{-\cos(200\pi t)}{200\pi} \right]_{-0.0025}^{0.0075} \right\}$$

$$= \frac{25}{\sqrt{2}} \left\{ \left[\frac{0-0}{200\pi} \right] + [0.0075 + 0.0025] + \left[\frac{-1-(-1)}{200\pi} \right] \right\}$$

$$- j\, \frac{25}{\sqrt{2}} \left\{ [0.0075 + 0.0025] - \frac{[-1-(-1)]}{200\pi} + \left[\frac{0-0}{200\pi} \right] \right\}$$

$$= \frac{25}{\sqrt{2}} [0.01 - j0.01]$$

$$= 0.1768 - j0.1768$$

$$= 0.25 \text{ at } -45^\circ \text{ (V)}$$

Since $v(t)$ is real then:

$$\tilde{C}'_{-1} = \tilde{C}'^{*}_{1}$$

$$= 0.1768 + j0.1768$$

$$= 0.25 \text{ at } 45^\circ \text{ (V)}$$

$$\tilde{C}'_2 = \frac{1}{T} \int_{t}^{t+T} v(t)\, e^{-j2\pi f_2 t}\, dt$$

$$= \frac{1}{0.02} \int_{-0.0025}^{0.0075} \sin\left(2\pi \frac{1}{0.02} t + \frac{2\pi\, 0.0025}{0.02} \right) e^{-j2\pi \frac{2}{0.02} t}\, dt$$

$$= 0.1061 \text{ at } -90^\circ \text{ (V)}$$

Since $v(t)$ is real then:

$$\tilde{C}'_{-2} = \tilde{C}'^{*}_{2} = 0.1061 \text{ at } 90^\circ \text{ (V)}$$

The amplitude and phase spectra are shown in Figure 2.18(b). The sum of the DC term, fundamental and second harmonic is shown in Figure 2.18(c). It is interesting to see that even with so few terms the Fourier series is a recognisable approximation to the half-wave rectified sinusoid. The total power in the DC, fundamental and second harmonic components is given by Parseval's theorem, equation (2.28), i.e.:

$$P = \sum_{n=-2}^{2} |\tilde{C}'_n|^2 \text{ (V}^2\text{)}$$

$$= 0.1061^2 + 0.25^2 + (1/\pi)^2 + 0.25^2 + 0.1061^2$$

$$= 0.2488 \text{ (V}^2\text{)}$$

Table 2.2 shows commonly encountered periodic waveforms and their corresponding Fourier series.

2.2.3 Conditions for existence, convergence and Gibb's phenomenon

The question might be asked: how do we know that it is possible to approximate $v(t)$ with a Fourier series and furthermore that adding further terms to the series continues to improve the approximation? Here we give the answer to this question without proof.

A function $v(t)$ has a Fourier series if the following conditions are met:

1. $v(t)$ contains a finite number of maxima and minima per period.
2. $v(t)$ contains a finite number of discontinuities per period.
3. $v(t)$ is absolutely integrable over one period, i.e.:

$$\int_{t}^{t+T} |v(t)| \, dt < \infty$$

The above conditions, called the Dirichlet conditions, are sufficient but not necessary. If a Fourier series does exist it converges (i.e. gets closer to $v(t)$ as more terms are added) at all points except points of discontinuity. Mathematically, this can be stated as follows:

$$\sum_{N} \text{series } |_{t_0} \rightarrow v(t_0) \text{ as } N \rightarrow \infty \text{ for all continuous points } t_0.$$

At points of discontinuity the series converges to the arithmetic mean of the function value on either side of the discontinuity, Figure 2.19(a), i.e.:

Table 2.2 *Fourier series of commonly occurring waveforms.*

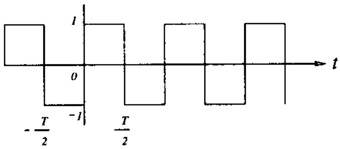
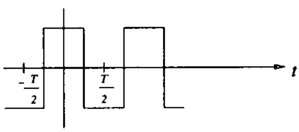
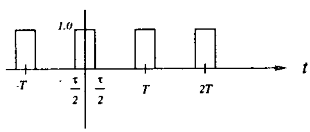
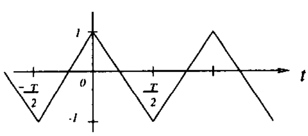
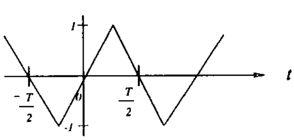
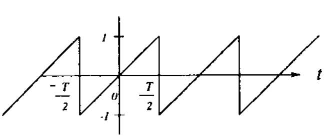
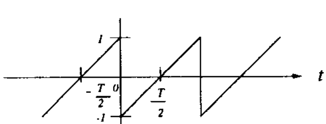
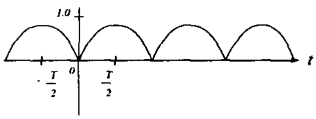
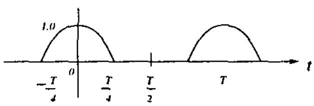
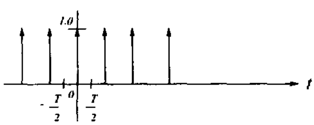
	$\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin\left[2\pi\frac{(2n-1)}{T}t\right]$
	$\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}\cos\left[2\pi\frac{(2n-1)}{T}t\right]$
	$\frac{\tau}{T}+\frac{2\tau}{T}\sum_{n=1}^{\infty}\mathrm{sinc}\left(\frac{n\tau}{T}\right)\cos\left(2\pi\frac{n}{T}t\right)$
	$\frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\left[2\pi\left(\frac{2n-1}{T}\right)t\right]$
	$\frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\sin\left[2\pi\left(\frac{2n-1}{T}\right)t\right]$
	$\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\left[2\pi\left(\frac{n}{T}\right)t\right]$
	$-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left[2\pi\left(\frac{n}{T}\right)t\right]$

Table 2.2 ctd. *Fourier series of commonly occurring waveforms.*

	$\frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos\left[2\pi\left(\frac{n}{T}\right)t\right]$
	$\frac{2}{\pi}\left\{\frac{1}{2}+\frac{\pi}{4}\cos\left[2\pi\left(\frac{1}{T}\right)t\right] - \sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1}\cos\left[2\pi\left(\frac{2n}{T}\right)t\right]\right\}$
	$\frac{1}{T}+\sum_{n=1}^{\infty}\frac{2}{T}\cos\left[2\pi\left(\frac{n}{T}\right)t\right]$

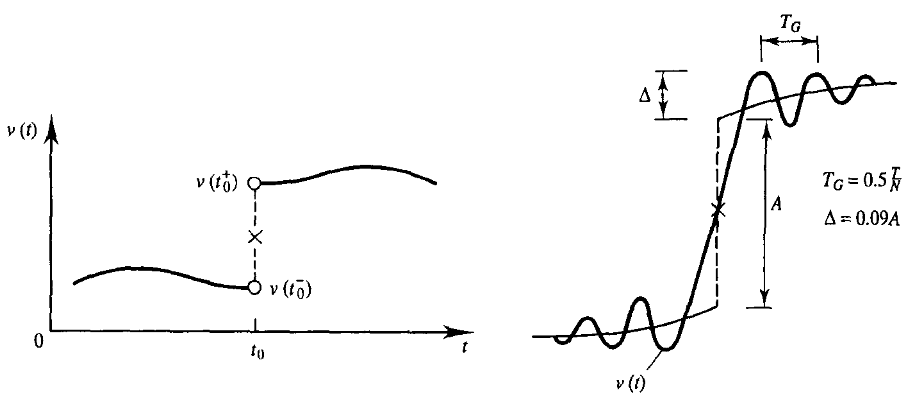
$$\sum_{N}\text{series}\,\big|_{t_0} \rightarrow \frac{v(t_0^-)+v(t_0^+)}{2} \text{ as } N\rightarrow\infty \text{ for all discontinuous points } t_0.$$

At points on either side of a discontinuity the series oscillates with a period T_G given by:

$$T_G = 0.5\,T/N \qquad (2.30(a))$$

where T is the period of $v(t)$ and N is the number of terms included in the series. The amplitude, Δ, of the overshoot on either side of the discontinuity is:

$$\Delta = 0.09A \qquad (2.30(b))$$



(a) Point of convergence (×) for Fourier series at a discontinuity (b) Gibb's ears on either side of discontinuity

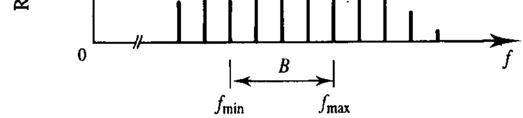
Figure 2.19 *Overshoot and undershoot of a truncated Fourier series at a point of discontinuity.*

where A is the amplitude of the discontinuity, Figure 2.19(b). The overshoot, Δ, does not decrease as N increases, the resulting spikes sometimes being known as 'Gibb's ears'.

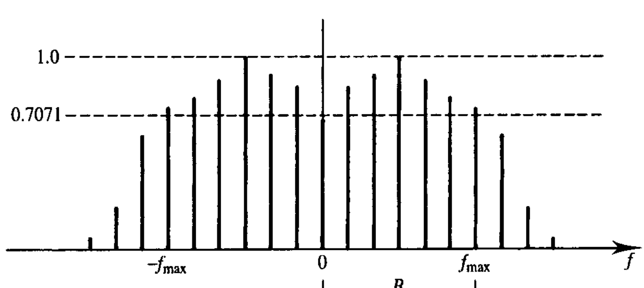
2.2.4 Bandwidth, rates of change, sampling and aliasing

The bandwidth, B, of a signal is defined as the difference (usually in Hz) between two nominal frequencies f_{max} and f_{min}. Loosely speaking f_{max} and f_{min} are, respectively, the frequencies above and below which the spectral components are assumed to be small. It is important to realise that these frequencies are often chosen using some fairly arbitrary rule, e.g. the frequencies at which spectral components have fallen to $1/\sqrt{2}$ of the peak spectral component. It would therefore be wrong to assume always that the frequency components of a signal outside its quoted bandwidth are negligible for all purposes, especially if the precise definition being used for B is vague or unknown.

The $1/\sqrt{2}$ definition of B is a common one and is *usually* implied if no other definition is explicitly given. It is normally called the half power or 3 dB bandwidth since the factor $1/\sqrt{2}$ refers to the voltage spectrum and $20 \log_{10}(1/\sqrt{2}) \approx -3$ dB. The 3 dB bandwidth of a periodic signal is illustrated in Figure 2.20(a). For baseband signals (i.e. signals with



(a) 3 dB bandwidth of a (bandpass) periodic signal



(b) 3 dB bandwidth of a (baseband) periodic signal shown on a double sided spectrum

Figure 2.20 *Definition of 3 dB signal bandwidth.*

significant spectral components all the way down to their fundamental frequency, f_1, or even DC) f_{min} is 0 Hz, *not* $-f_{max}$. This is important to remember when considering two sided spectra. The physical bandwidth is measured using positive frequencies or negative frequencies only, not both, Figure 2.20(b).

In general, if a signal has no significant spectral components above f_H then it cannot change appreciably on a time scale much shorter than about $1/(8f_H)$. (This corresponds to one eighth of a period of the highest frequency sinusoid present in the signal, Figure 2.21.) A corollary of this is that signals with large rates of change must have high values of f_H. A rectangular pulse stream, for example, contains changes which occur (in principle) infinitely quickly. This implies that it must contain spectral components with infinite frequency. (In practice, of course, such pulse streams are, at best, only approximately rectangular and therefore their spectra can be essentially bandlimited.)

Sampling refers to the process of recording the values of a signal or waveform at (usually) regularly spaced instants of time. A schematic diagram of how this might be achieved is shown in Figure 2.22. It is a surprising fact that if a signal having no spectral components with frequencies above f_H is sampled rapidly enough then the original, continuous, signal can, in principle, be reconstructed from its samples *without error*. The

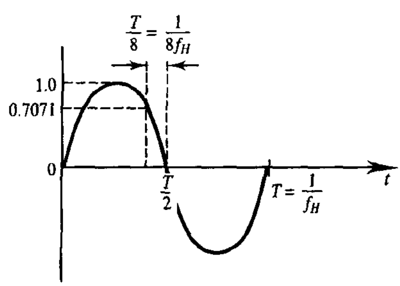


Figure 2.21 *Illustration of minimum time required for appreciable change of signal amplitude.*

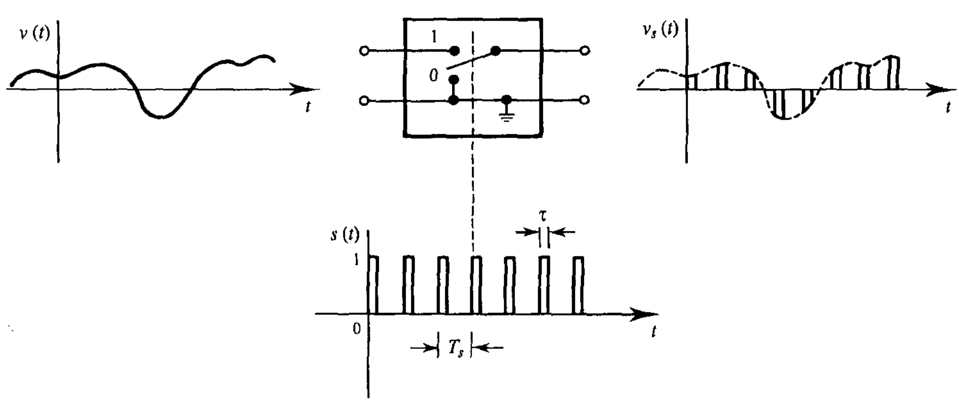


Figure 2.22 *Schematic illustration of sampling.*

minimum sampling rate or frequency, f_s, needed to achieve such ideal reconstruction is related to f_H by:

$$f_s \geq 2f_H \tag{2.31}$$

Equation (2.31) is called Nyquist's sampling theorem and is of central importance to digital communications. It will be discussed more rigourously in Chapter 5. Here, however, it is sufficient to demonstrate its reasonableness as follows.

Figure 2.23(a) shows a sinusoid which represents the highest frequency spectral component in a certain waveform. The sinusoid is sampled in accordance with equation (2.31), i.e. at a rate higher than twice its frequency. (When $f_s > 2f_H$ the signal is said to be *oversampled.*) Nyquist's theorem essentially says that there is one, and only one, sinusoid which can be drawn through the given sample points. Figure 2.23(b) shows the same sinusoid sampled at a rate $f_s = 2f_H$. (This might be called critical, or Nyquist rate, sampling.) There is still only one frequency of sine wave which can be drawn through

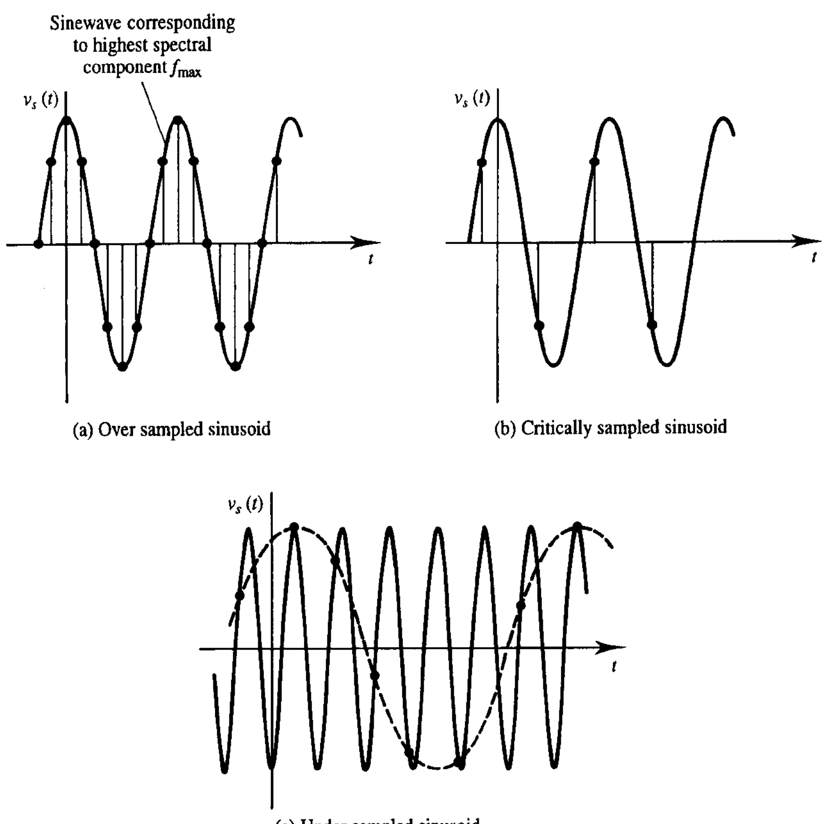


Figure 2.23 *Demonstration of the sampling theorem and alias frequency.*

the samples. Figure 2.23(c) shows a sinusoid which is now *undersampled* (i.e. $f_s < 2f_H$). The samples could be (and usually are) interpreted as belonging to a sinusoid (shown dotted) of lower frequency than that to which they actually belong. The mistaken identity of the frequency of an undersampled sinusoid is called *aliasing* since the sinusoid inferred from the samples appears under the alias of a new and incorrect frequency. Aliasing is explained more fully later (section 5.3.3) with the aid of frequency domain concepts.

2.3 Transient signals

Signals are said to be transient if they are essentially localised in time. This obviously includes time limited signals which have a well defined start and stop time and which are zero outside the start-stop time interval. Signals with no start time, stop time, or either, are usually also considered to be transient, providing they tend to zero as time tends to $\pm\infty$ and contain finite total energy. Since the power of such signals averaged over all time is zero they are sometimes called *energy* signals. Since transient signals are not periodic they cannot be represented by an ordinary Fourier series. A related but more general technique, namely Fourier transformation, can, however, be used to find a frequency domain, or spectral, description of such signals.

2.3.1 Fourier transforms

The traditional way of approaching Fourier transforms is to treat them as a limiting case of a periodic signal Fourier series as the period, T, tends to infinity. Consider Figure 2.24. The waveform in this figure is periodic and pulsed with interpulse spacing, T_g. The amplitude and phase spectra of $v(t)$ are shown (schematically) in Figure 2.25(a) and (b) respectively. They are discrete (since $v(t)$ is periodic), have even and odd symmetry respectively (since $v(t)$ is real) and have line spacing $1/T$ Hz. If the interpulse spacing is now allowed to grow without limit (i.e. $T_g \rightarrow \infty$) then it follows that:

1. Period, $T \rightarrow \infty$.
2. Spacing of spectral lines, $1/T \rightarrow 0$.
3. The discrete spectrum becomes continuous (as $V(f)$ is defined at all points).
4. The signal becomes aperiodic (since only one pulse is left between $t = -\infty$ and $t = \infty$).

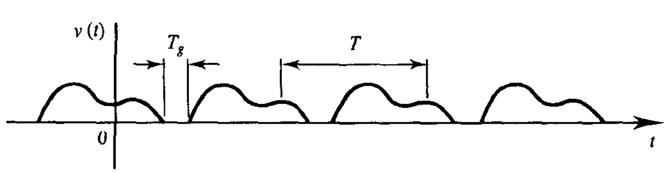


Figure 2.24 *Pulsed, periodic waveform.*

As the spectral lines become infinitesimally closely spaced the discrete quantities in the Fourier series:

$$v(t) = \sum_{n=-\infty}^{\infty} \tilde{C}'_n \, e^{j2\pi f_n t} \tag{2.32}$$

become continuous, i.e.:

$$f_n \rightarrow f \quad \text{(Hz)}$$

$$\tilde{C}'_n(f_n) \rightarrow V(f)\, df \quad \text{(V)}$$

$$\sum_{n=-\infty}^{\infty} \rightarrow \int_{-\infty}^{\infty}$$

Since $\tilde{C}'_n(f_n)$, and therefore $V(f)\, df$, have units of V then $V(f)$ has units of V/Hz. $V(f)$ is called a *voltage spectral density*. The resulting 'continuous series', called an *inverse* Fourier transform, is:

$$v(t) = \int_{-\infty}^{\infty} V(f)\, e^{j2\pi ft}\, df \tag{2.33}$$

The converse formula, equation (2.25), which gives the (complex) Fourier coefficients for a Fourier series, is:

$$\tilde{C}'_n = \frac{1}{T} \int_{t-T/2}^{t+T/2} v(t)\, e^{-j2\pi f_n t}\, dt \tag{2.34}$$

If this is generalised in the same way as equation (2.32) by letting $T \rightarrow \infty$ then $\tilde{C}'_n \rightarrow 0$ for all n and $v(t)$. This problem can be avoided by calculating $T\tilde{C}'_n$ instead. In this case:

$$f_n \rightarrow f$$

$$T\tilde{C}'_n \rightarrow V(f)$$

$$t \pm \frac{T}{2} \rightarrow \pm\infty$$

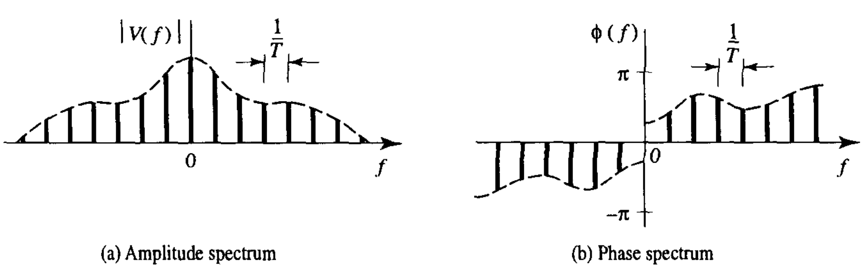


Figure 2.25 *Voltage spectrum of a pulsed, periodic, waveform.*

(Note that $T\tilde{C}'_n$ and $V(f)$ have units of V s or equivalently V/Hz as required.) The *forward* Fourier transform therefore becomes:

$$V(f) = \int_{-\infty}^{\infty} v(t)\, e^{-j2\pi ft}\, dt \tag{2.35}$$

Equation (2.35) can be interpreted as finding that part of $v(t)$ which is identical to $e^{j2\pi ft}$. This is a cisoid (or rotating phasor) with frequency f and amplitude $V(f)\, df$ (V). For real signals there will be a conjugate cisoid rotating in the opposite sense located at $-f$. This pair of cisoids together constitute a sinusoid of frequency f and amplitude $2V(f)\, df$ (V). A one sided amplitude spectrum can thus be formed by folding the negative frequency components of the two sided spectrum, defined by equation (2.35), onto the positive frequencies and adding.

Sufficient conditions for the existence of a Fourier transform are similar to those for a Fourier series. They are:

1. $v(t)$ contains a finite number of maxima and minima in any finite time interval.
2. $v(t)$ contains a finite number of *finite* discontinuities in any finite time interval.
3. $v(t)$ must be absolutely integrable, i.e.:

$$\int_{-\infty}^{\infty} |v(t)|\, dt < \infty$$

2.3.2 Practical calculation of Fourier transforms

As with Fourier series simplification of practical calculations is possible if certain symmetries are present in the function being transformed. This is best explained by splitting the Fourier transform into cosine and sine transforms as follows:

$$\begin{aligned} V(f) &= \int_{-\infty}^{\infty} v(t)\, e^{-j2\pi ft}\, dt \\ &= \int_{-\infty}^{\infty} v(t) \cos 2\pi ft\, dt - j \int_{-\infty}^{\infty} v(t) \sin 2\pi ft\, dt \end{aligned} \tag{2.36}$$

The first term in the second line of equation (2.36) is made up of cosine components only. It therefore corresponds to a component of $v(t)$ which has *even* symmetry. Similarly the second term is made up of sine components only and therefore corresponds to an *odd* component of $v(t)$, i.e.:

$$V(f) = V(f)|_{even\ v(t)} + jV(f)|_{odd\ v(t)} \tag{2.37(a)}$$

where:

$$V(f)|_{even\ v(t)} = \int_{-\infty}^{\infty} v(t) \cos 2\pi ft\, dt \tag{2.37(b)}$$

and:

$$V(f)|_{odd\ v(t)} = -\int_{-\infty}^{\infty} v(t) \sin 2\pi ft\ dt \qquad (2.37(c))$$

It follows that if $v(t)$ is purely even (and real) then:

$$V(f) = 2\int_{0}^{\infty} v(t) \cos 2\pi ft\ dt \qquad (2.38(a))$$

Conversely, if $v(t)$ is purely odd (and real) then:

$$V(f) = -2j\int_{0}^{\infty} v(t) \sin 2\pi ft\ dt \qquad (2.38(b))$$

That any function can be split into odd and even parts is easily demonstrated, as follows:

$$v(t) = \frac{v(t) + v(-t)}{2} + \frac{v(t) - v(-t)}{2} \qquad (2.39)$$

The first term on the right hand side of equation (2.39) is, by definition, even and the second term is odd. A summary of symmetry properties relevant to the calculation of Fourier transforms is given in Table 2.3 [after Bracewell].

Table 2.3 *Symmetry properties of Fourier transforms.*

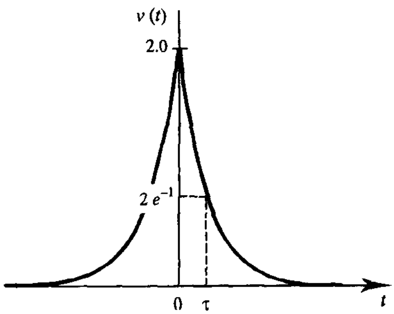
Function	*Transform*
real and even	real and even
real and odd	imaginary and odd
imaginary and even	imaginary and even
imaginary and odd	real and odd
complex and even	complex and even
complex and odd	complex and odd
real and asymmetrical	complex and Hermitian
imaginary and asymmetrical	complex and antiHermitian
real even plus imaginary odd	real
real odd plus imaginary even	imaginary
even	even
odd	odd

EXAMPLE 2.3

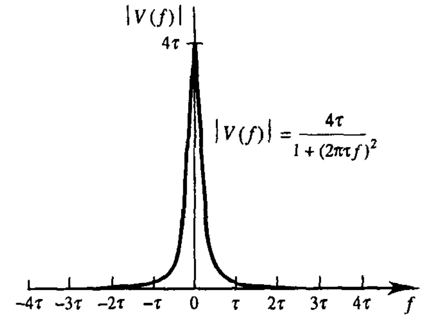
Find and sketch the amplitude and phase spectrum of the transient signal $v(t) = 2e^{-|t|/\tau}$ (V) shown in Figure 2.26(a).

Since $v(t)$ is real and even:

$$V(f) = 2\int_{0}^{\infty} v(t) \cos 2\pi ft\ dt$$



(a) Double sided exponential function $v(t) = 2\,e^{-|t|/\tau}$



(b) Double sided amplitude spectrum of signal in Example 2.3

Figure 2.26 *Double sided exponential function and corresponding amplitude spectrum.*

$$= 4 \int_0^{\infty} e^{-|t|/\tau} \cos 2\pi f t \, dt$$

Using the standard integral:

$$\int e^{ax} \cos bx \, dx = \frac{e^{ax}(a \cos bx + b \sin bx)}{a^2 + b^2}$$

$$V(f) = 4 \left[\frac{e^{-(1/\tau)t} [- 1/\tau \cos 2\pi f t + 2\pi f \sin 2\pi f t]}{(1/\tau)^2 + (2\pi f)^2} \right]_0^{\infty}$$

$$= 4 \left[\frac{0 - [-1/\tau + 0]}{(1/\tau)^2 + (2\pi f)^2} \right] = \frac{4\tau}{1 + (2\pi\tau f)^2} \quad \text{(V/Hz)}$$

$|V(f)|$ is sketched in Figure 2.26(b) and since $V(f)$ is everywhere real and positive then $v(t)$ has a null phase spectrum.

2.3.3 Fourier transform pairs

The Fourier transform (for transient functions) and Fourier series (for periodic functions) provide a link between two quite different ways of describing signals. The more familiar description is the conventional time plot such as would be seen on an oscilloscope display. Applying the Fourier transform results in a frequency plot (amplitude and phase). These two descriptions are equivalent in the sense that there is one, and only one, amplitude and phase spectrum pair for each possible time plot. Given a complete time domain description, therefore, the frequency domain description can be obtained exactly and vice versa.

Comprehensive tables of Fourier transform pairs have been compiled by many authors. Table 2.4 lists some common Fourier transform pairs. The notation used here for several of the functions which occur frequently in communications engineering is included at the front of this text following the list of principal symbols. Owing to its central importance in digital communications, the Fourier transform of the rectangular function is derived from first principles below. Later the impulse function is defined as a limiting case of the rectangular pulse. The Fourier transform of the impulse is then shown to be a constant in amplitude and linear in phase.

Table 2.4 *Fourier transform pairs.*

Function	$x(t)$	$X(f)$
Rectangle of unit width	$\Pi(t)$	$\text{sinc}(f)$
Delayed rectangle of width τ	$\Pi\left(\frac{t-T}{\tau}\right)$	$\tau \,\text{sinc}(\tau f)e^{-j\omega T}$
Triangle of base width 2τ	$\Lambda\left(\frac{t}{\tau}\right)$	$\tau \,\text{sinc}^2(\tau f)$
Gaussian	$e^{-\pi(t/\tau)^2}$	$\tau e^{-\pi(\tau f)^2}$
One sided exponential	$u(t)\, e^{-t/\tau}$	$\frac{\tau}{1 + j2\pi\tau f}$
Two sided exponential	$e^{-\lvert t\rvert/\tau}$	$\frac{2\tau}{1 + (2\pi\tau f)^2}$
sinc	$\text{sinc}(2f_\chi t)$	$\frac{1}{2f_\chi}\Pi\left(\frac{f}{2f_\chi}\right)$
Constant	1	$\delta(f)$
Phasor	$e^{j(\omega_c t + \phi)}$	$e^{j\phi}\delta(f - f_c)$
sine wave	$\sin(\omega_c t + \phi)$	$\frac{1}{2j}\left[e^{j\phi}\delta(f - f_c) - e^{-j\phi}\delta(f + f_c)\right]$
cosine wave	$\cos(\omega_c t + \phi)$	$\frac{1}{2}\left[e^{j\phi}\delta(f - f_c) + e^{-j\phi}\delta(f + f_c)\right]$
Impulse	$\delta(t - T)$	$e^{-j\omega T}$
Sampling	$\sum_{k=-\infty}^{\infty} \delta(t - kT_s)$	$f_s \sum_{n=-\infty}^{\infty} \delta(f - nf_s)$
Signum	$\text{sgn}(t)$	$\frac{1}{j\pi f}$
Heaviside step	$u(t)$	$\frac{1}{2}\delta(f) + \frac{1}{j2\pi f}$

Fourier transform of a rectangular pulse

The unit rectangular pulse, Figure 2.27(a), is represented here using the notation $\Pi(t)$ and is defined by:

$$\Pi(t) \triangleq \begin{cases} 1.0, & \lvert t\rvert < ½ \\ 0.5, & \lvert t\rvert = ½ \\ 0, & \lvert t\rvert > ½ \end{cases} \tag{2.40}$$

The voltage spectrum, $V_\Pi(f)$, of this pulse is given by its Fourier transform, i.e.:

$$\begin{aligned} V_\Pi(f) &= \int_{-\infty}^{\infty} \Pi(t)\, e^{-j2\pi ft}\, dt \\ &= \int_{-1/2}^{1/2} e^{-j2\pi ft}\, dt \\ &= \left[\frac{e^{-j2\pi ft}}{-j2\pi f} \right]_{-1/2}^{1/2} = \frac{1}{j2\pi f} [e^{j\pi f} - e^{-j\pi f}] \\ &= \frac{j2 \sin(\pi f)}{j2\pi f} = \frac{\sin(\pi f)}{\pi f} \end{aligned} \tag{2.41}$$

The function $\text{sinc}(x)$ is defined by:

$$\text{sinc}(x) \triangleq \frac{\sin(\pi x)}{\pi x} \tag{2.42}$$

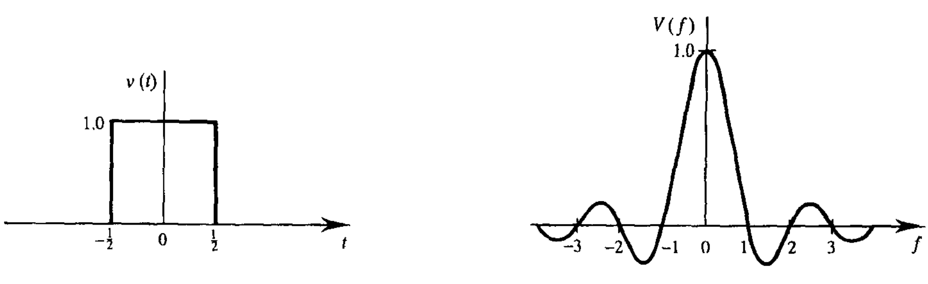
which means that the unit rectangular pulse and unit sinc function form a Fourier transform pair:

$$\Pi(t) \overset{FT}{\Leftrightarrow} \text{sinc}(f) \tag{2.43(a)}$$

The $\text{sinc}(f)$ function is shown in Figure 2.27(b). Whilst in this case the voltage spectrum can be plotted as a single curve, in general the voltage spectrum of a transient signal is complex and must be plotted either as amplitude and phase spectra or as inphase and quadrature spectra. The amplitude and phase spectra corresponding to Figure 2.27(b) are shown in Figure 2.28(a) and (b).

It is left to the reader to show that the (complex) voltage spectrum of $\Pi[(t-T)/\tau]$ where T is the location of the centre of the pulse and τ is its width is given by:

$$\Pi\left(\frac{t-T}{\tau}\right) \overset{FT}{\Leftrightarrow} \tau\, \text{sinc}(\tau f)\, e^{-j2\pi fT} \tag{2.43(b)}$$



(a) Unit rectangular pulse, $\Pi(t)$ (b) Fourier transform of unit rectangular pulse centred on $t = 0$

Figure 2.27 *Unit rectangular pulse and corresponding Fourier transform.*

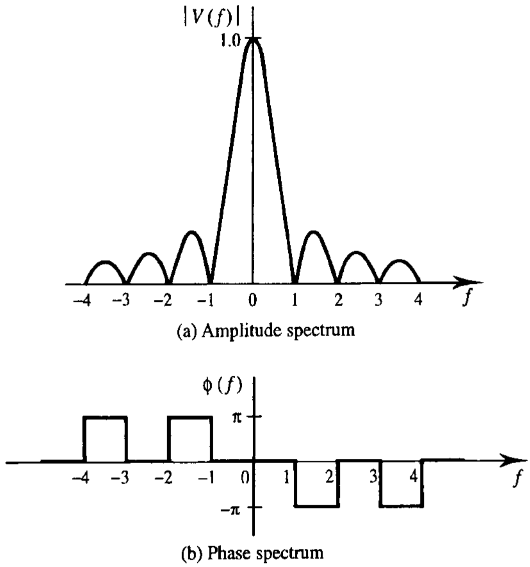


Figure 2.28 *Voltage spectrum of unit rectangular pulse shown in Figure 2.27(a).*

The impulse function and its Fourier transform

Consider a tall, narrow, rectangular voltage pulse of width τ seconds and amplitude $1/\tau$ V occurring at time $t = T$, Figure 2.29. The area under the pulse (sometimes called its strength) is clearly 1.0 V s. The impulse function (also called the Dirac delta function) can be defined as the limit of this rectangular pulse as τ tends to zero, i.e.:

$$\delta(t - T) = \lim_{\tau \to 0} \left(\frac{1}{\tau}\right) \Pi\left(\frac{t - T}{\tau}\right) \tag{2.44}$$

This idea is illustrated in Figure 2.30. Whatever the value of τ the strength of the pulse remains unity. Mathematically the impulse might be described by:

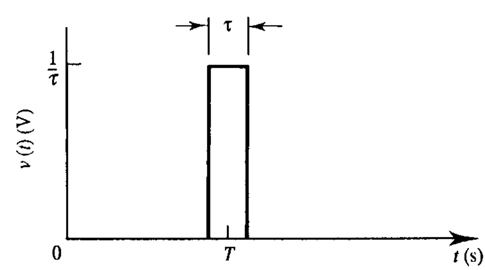


Figure 2.29 *Tall, narrow rectangular pulse of unit strength.*

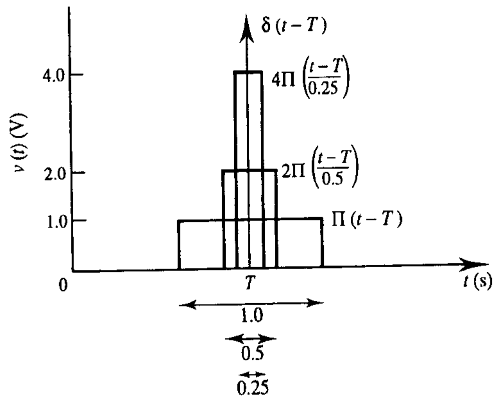


Figure 2.30 *Development of unit strength impulse, $\delta(t - T)$, as a limit of a sequence of unit strength rectangular pulses.*

$$\delta(t - T) = \begin{cases} \infty, & t = T \\ 0, & t \neq T \end{cases} \tag{2.45(a)}$$

$$\int_{-\infty}^{\infty} \delta(t - T)\, dt = \int_{T^-}^{T^+} \delta(t - T)\, dt = 1.0 \tag{2.45(b)}$$

More strictly the impulse is *defined* by its sampling, or *sifting*, property under integration, i.e.:

$$\int_{-\infty}^{\infty} \delta(t - T)\, f(t)\, dt = f(T) \tag{2.46(a)}$$

That equation (2.46(a)) is consistent with equations (2.45) is easily shown as follows:

$$\begin{aligned} \int_{-\infty}^{\infty} \delta(t - T)\, f(t)\, dt &= \int_{T^-}^{T^+} \delta(t - T)\, f(t)\, dt \\ &= \int_{T^-}^{T^+} \delta(t - T)\, f(T)\, dt \\ &= f(T) \int_{T^-}^{T^+} \delta(t - T)\, dt = f(T) \end{aligned} \tag{2.46(b)}$$

Notice that if we insist that the strength of the impulse has units of Vs, i.e. its amplitude has units of V, then the sampled quantity, $f(T)$, in equations (2.46) would have units of V^2s (or joules in 1 Ω). In view of this the impulse is usually taken to have an amplitude measured in s^{-1} (i.e. to have dimensionless strength). This can be reconciled with an

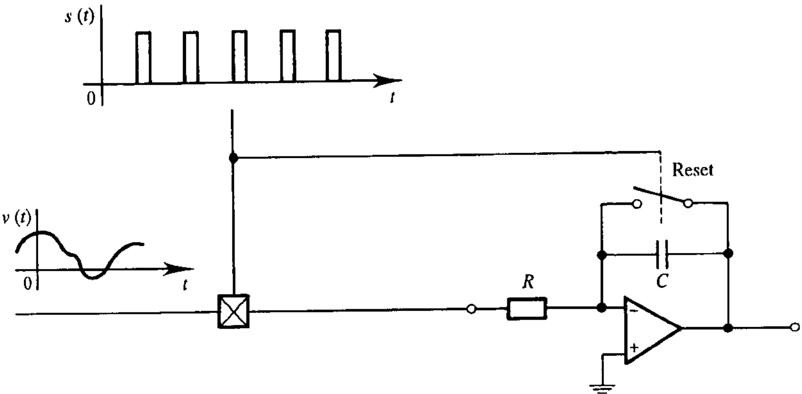


Figure 2.31 *Hypothetical sampling system reconciling physical units of impulse strength (Vs) with units of sampled signal (V).*

equivalent physical implementation of sampling using tall, narrow pulses, a multiplier and integrator by associating dimensions of V^{-1} with the multiplier (required for its output to have units of V) and dimensions of s^{-1} with the integrator (required for its output also to have units of V). Such an implementation, shown in Figure 2.31, is not, of course, used in practical sampling circuits.

As a rectangular pulse gets narrower its Fourier transform (which is a sinc function) gets wider, Figure 2.32. This reciprocal width relationship is a general property of all Fourier transform pairs. Using equation (2.43(b)) it can be seen that as $\tau \to 0$ then $\tau f \to 0$ and $\text{sinc}(\tau f) \to 1.0$. It follows that:

$$\lim_{\tau \to 0} \text{FT}\left\{ \frac{1}{\tau} \Pi\left(\frac{t-T}{\tau} \right) \right\} = e^{-j2\pi fT} \tag{2.47(a)}$$

i.e.:

$$\delta(t-T) \overset{\text{FT}}{\Leftrightarrow} e^{-j2\pi fT} \tag{2.47(b)}$$

For an impulse occurring at the origin this reduces to:

$$\delta(t) \overset{\text{FT}}{\Leftrightarrow} 1.0 \tag{2.47(c)}$$

The amplitude spectrum of an impulse function is therefore a constant (measured in V/Hz if $\delta(t)$ has units of V). Such a spectrum is sometimes referred to as white, since all frequencies are present in equal quantities. This is analogous to white light. (From a strict mathematical point of view the impulse function, *as represented here*, does not have a Fourier transform owing to the infinite discontinuity which it contains. The impulse and constant in equation (2.47(c)) can be approximated so closely by tall, thin rectangular pulses and broad sinc pulses, however, that the limiting forms need not be challenged. In any event, if desired, the impulse can be derived as the limiting form of other pulse shapes which contain no discontinuity.)

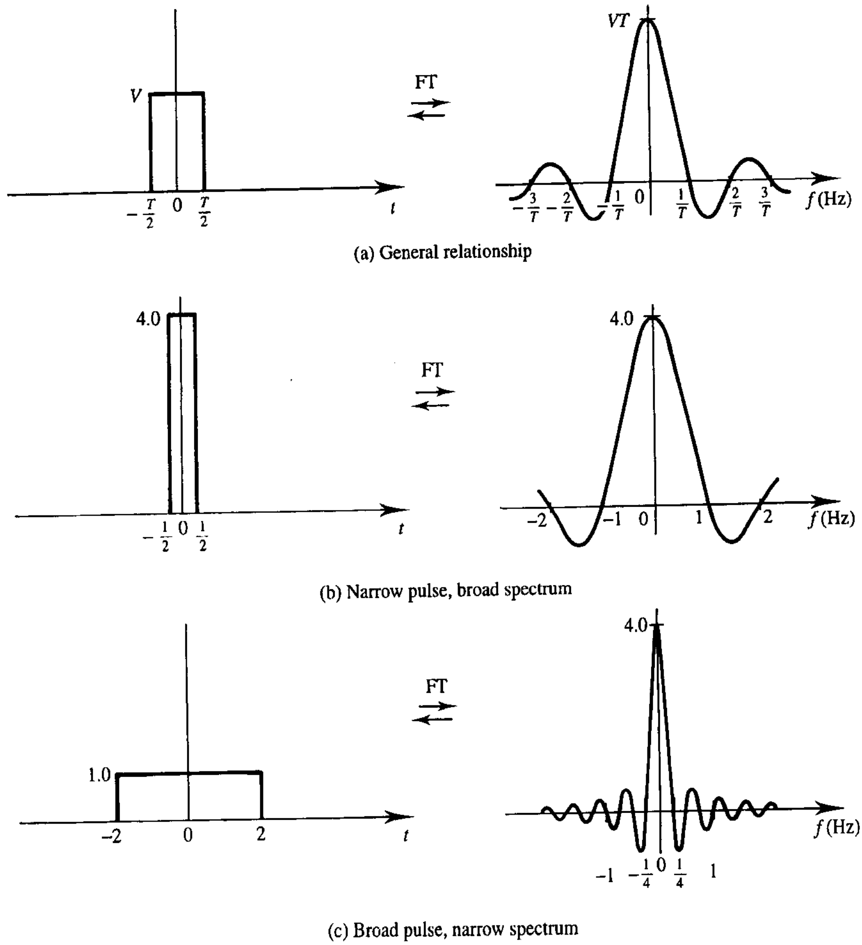


Figure 2.32 *Inverse width relationship between Fourier transform pairs.*

2.3.4 Fourier transform theorems and convolution

Since signals can be fully described in either the time or frequency domain it follows that any operation on a signal in one domain has a precisely equivalent operation in the other domain. A list of equivalent operations on $v(t)$ and its transform, $V(f)$, is given in the form of a set of theorems in Table 2.5. Most of the operations in this list (addition, multiplication, differentiation, integration), whether applied to the functions themselves or their arguments, will be familiar. One operation, namely convolution, may be unfamiliar, however, and is therefore described below.

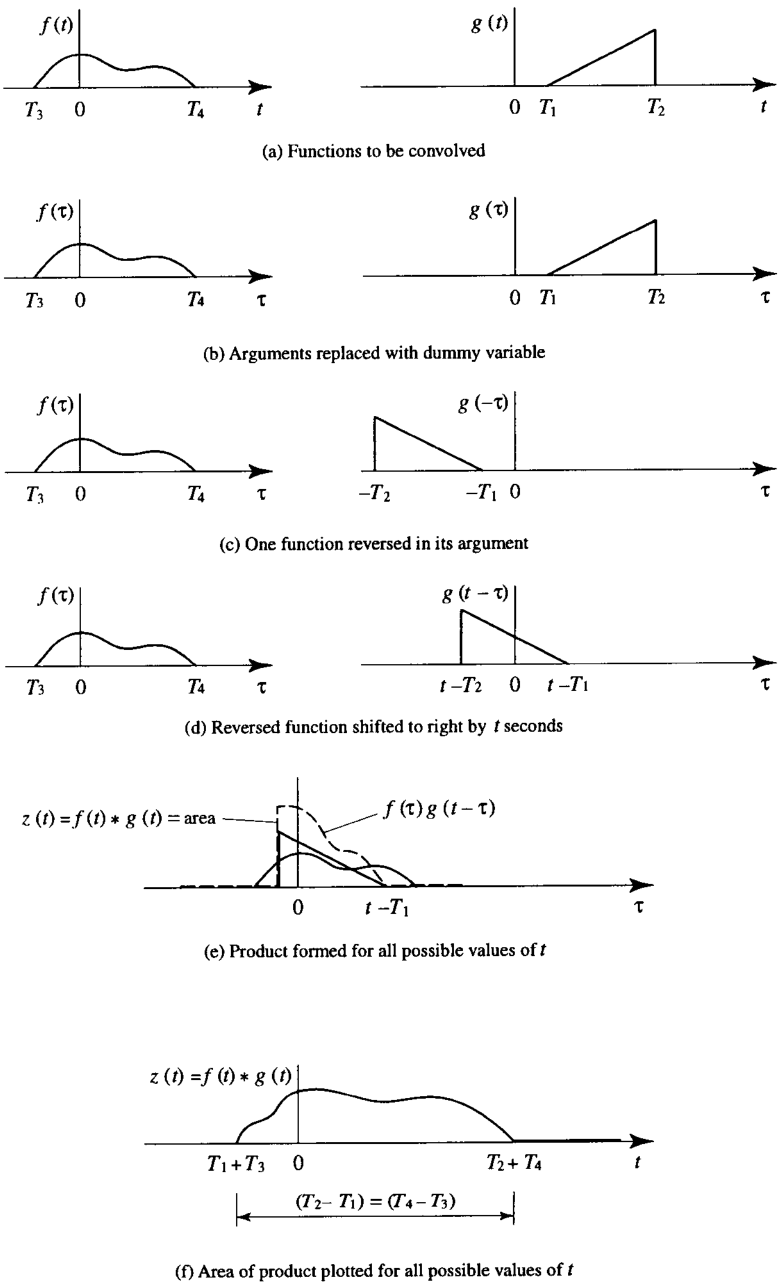


Figure 2.33 *Graphical illustration of time convolution.*

Table 2.5 *Fourier transform theorems.*

Linearity	$av(t) + bw(t)$	$aV(f) + bW(f)$
Time delay	$v(t - T)$	$V(f)e^{-j\omega T}$
Change of scale	$v(at)$	$\frac{1}{\lvert a\rvert} V\left(\frac{f}{a}\right)$
Time reversal	$v(-t)$	$V(-f)$
Time conjugation	$v^*(t)$	$V^*(-f)$
Frequency conjugation	$v^*(-t)$	$V^*(f)$
Duality	$V(t)$	$v(-f)$
Frequency translation	$v(t)e^{j\omega_c t}$	$V(f - f_c)$
Modulation	$v(t)\cos(\omega_c t + \phi)$	$\frac{1}{2}\left[e^{j\phi}V(f - f_c) + e^{-j\phi}V(f + f_c)\right]$
Time differentiation	$\frac{d^n}{dt^n} v(t)$	$(j2\pi f)^n V(f)$
Integration (1)	$\int_{-\infty}^{t} v(t')dt'$	$(j2\pi f)^{-1}V(f) + \frac{1}{2}V(0)\,\delta(f)$
Integration (2)	$\int_{0}^{t} v_e(t')\,dt' + \int_{-\infty}^{t} v_o(t')\,dt'$	$(j2\pi f)^{-1}V(f)$
Convolution	$v(t) * w(t)$	$V(f)W(f)$
Multiplication	$v(t)w(t)$	$V(f) * W(f)$
Frequency differentiation	$t^n v(t)$	$(-j2\pi)^{-n} \frac{d^n}{df^n} V(f)$

DC value	$V(0) = \int_{-\infty}^{\infty} v(t)\,dt$
Value at the origin	$v(0) = \int_{-\infty}^{\infty} V(f)\,df$
Integral of a product	$\int_{-\infty}^{\infty} v(t)w^*(t)\,dt = \int_{-\infty}^{\infty} V(f)W^*(f)\,df$

Convolution is normally denoted by * although ⊗ is also sometimes used. Applied to two time function $z(t) = f(t) * g(t)$ is defined by:

$$z(t) = \int_{-\infty}^{\infty} f(\tau)\, g(t - \tau)\, d\tau \tag{2.48(a)}$$

Figure 2.33 illustrates time convolution graphically. It can be thought of as a five step process:

1. The arguments of the functions to be convolved are replaced with a dummy variable (in this case τ), Figure 2.33(b).
2. One of the functions (arbitrarily chosen here to be $g(\tau)$) is reversed in its argument (i.e. reflected about $\tau = 0$) giving $g(-\tau)$, Figure 2.33(c).

3. A variable time shift, t, is introduced into the argument of the reflected function giving $g(t - \tau)$. This is a version of $g(-\tau)$ shifted to the right by t s, Figure 2.33(d).
4. The product function $f(\tau)g(t - \tau)$ is formed for every possible value of t, Figure 2.33(e). (This function changes continuously as t varies and $g(t - \tau)$ slides across $f(\tau)$.)
5. The area under the product function is calculated (by integrating) for every value of t.

There are several important points to note about convolution:

1. The convolution integral, equation (2.48(a)), is sometimes called the superposition integral.
2. It is simply convention which dictates that dummy variables are used so that the result can be expressed as a function of the argument of f and g. There is no reason, in principle, why the alternative definition:

$$z(\tau) = \int_{-\infty}^{\infty} f(t)\, g(\tau - t)\, dt \tag{2.48(b)}$$

should not be used.

3. Convolution is not restricted to the time domain. It can be applied to functions of any variable, for example frequency, f, i.e.:

$$Z(f) = F(f) * G(f) = \int_{-\infty}^{\infty} F(\phi)G(f - \phi)\, d\phi \tag{2.49(a)}$$

or space, x, i.e.:

$$z(x) = f(x) * g(x) = \int_{-\infty}^{\infty} f(\lambda)\, g(x - \lambda)\, d\lambda \tag{2.49(b)}$$

4. The unitary operator for convolution is the impulse function $\delta(t)$ since convolution of $f(t)$ with $\delta(t)$ leaves $f(t)$ unchanged, i.e.:

$$f(t) * \delta(t) = \int_{-\infty}^{\infty} f(\tau)\, \delta(t - \tau)\, d\tau = f(t) \tag{2.50(a)}$$

(This follows directly from the sampling property of $\delta(t)$ under integration.)

5. Convolution in the time domain corresponds to multiplication in the frequency domain and vice versa.
6. Convolution is commutative, associative and distributive, i.e.:

$$f * g = g * f \tag{2.50(b)}$$

$$f * (g * h) = (f * g) * h \tag{2.50(c)}$$

$$f * (g + h) = f * g + f * h \tag{2.50(d)}$$

7. The derivative of a convolution is the derivative of one function convolved with the other, i.e.:

$$\frac{d}{dt}[v(t) * w(t)] = \frac{dv(t)}{dt} * w(t) = v(t) * \frac{dw(t)}{dt} \tag{2.50(e)}$$

EXAMPLE 2.4

Convolve the two transient signals, $x(t) = \Pi[(t-1)/2] \sin \pi t$ and $h(t) = 2\,\Pi[(t-2)/2])$, shown in Figure 2.34(a).

For $t < 1$ the picture of the convolution process looks like Figure 2.34(b), i.e. $y(t) = x(t) * h(t) = 0$.

For $1 \le t \le 3$ the picture looks like Figure 2.34(c) and:

$$\begin{aligned} y(t) &= \int_0^{t-1} 2\sin(\pi\tau)\,d\tau \\ &= 2\left[\frac{-\cos\pi\tau}{\pi}\right]_0^{t-1} \\ &= \frac{2}{\pi}[1 - \cos\pi(t-1)] \end{aligned}$$

For $3 \le t \le 5$ the picture looks like Figure 2.34(d):

$$\begin{aligned} y(t) &= \int_{t-3}^{2} 2\sin(\pi\tau)\,d\tau \\ &= 2\left[\frac{-\cos(\pi\tau)}{\pi}\right]_{t-3}^{2} \\ &= \frac{2}{\pi}[\cos\pi(t-3) - 1] \end{aligned}$$

For $t > 5$ there is no overlapping of the two functions; therefore $y(t) = 0$. Figure 2.34(e) shows a sketch of $y(t)$. Note that the convolved output signal has a duration of 4 time units (i.e. the sum of the durations of the input signals) and, when the two input signals exactly overlap at $t = 3$, the output is 0 as expected.

EXAMPLE 2.5

Convolve the function $\Pi(t - ½)$ with itself and show that the Fourier transform of the result is the square of the Fourier transform of $\Pi(t - ½)$.

$$z(t) = \Pi(t - \tfrac{1}{2}) * \Pi(t - \tfrac{1}{2}) = \int_{-\infty}^{\infty} \Pi(\tau - \tfrac{1}{2})\,\Pi(t - \tau - \tfrac{1}{2})\,d\tau$$

For $t < 0$, $z(t) = 0$ (by inspection), Figure 2.35(a).

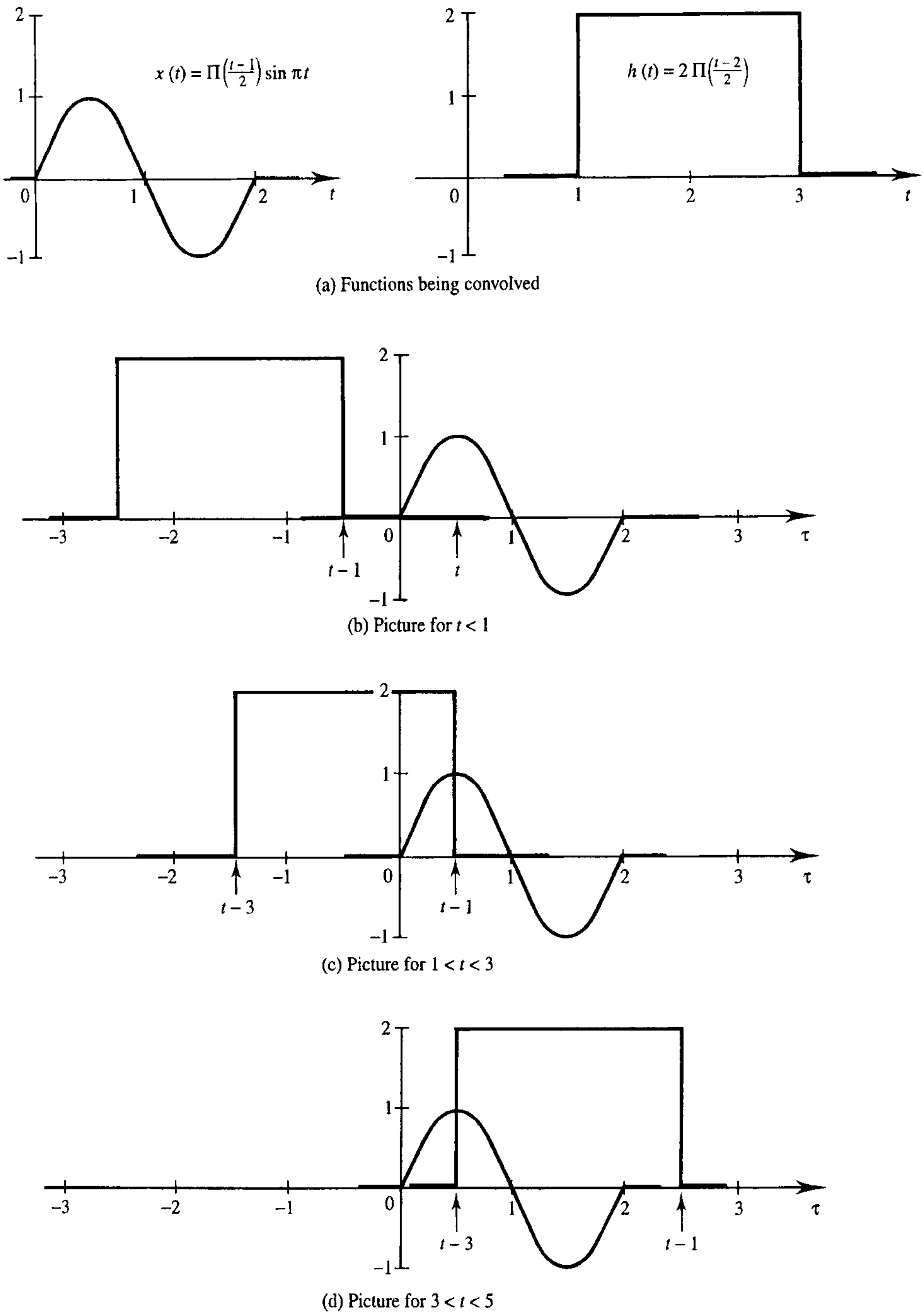
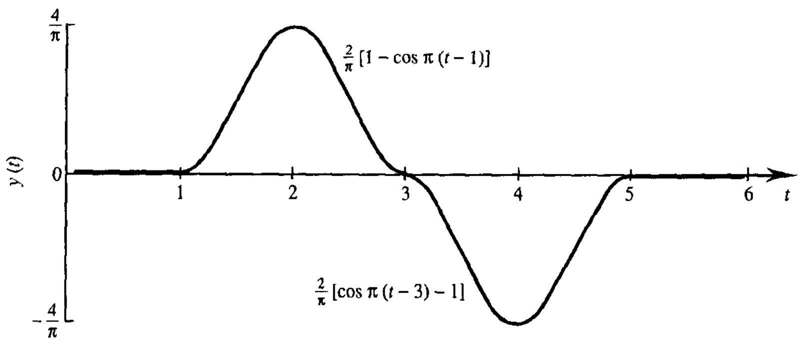


Figure 2.34 *Graphical illustration of time convolution.*



(e) Sketch of $y(t) = x(t) * h(t)$

Figure 2.34-ctd. *Graphical illustration of time convolution.*

$$\text{For } 0 < t < 1,\quad z(t) = \int_0^t (1 \times 1)\, d\tau \quad \text{(Figure 2.35(b))}$$

$$= [\,\tau\,]_0^t = t$$

$$\text{For } 1 < t < 2,\quad z(t) = \int_{t-1}^1 (1 \times 1)\, d\tau \quad \text{(Figure2.35(c))}$$

$$= [\,\tau\,]_{t-1}^1 = 2 - t$$

For $t > 2$, $z(t) = 0$ (by inspection), Figure 2.35(d).

Figure 2.35(e) shows a sketch of $z(t)$. This function, for obvious reasons, is called the triangular function which, if centred on $t = 0$, is denoted by $\Lambda(t)$. (Note that the absolute width of

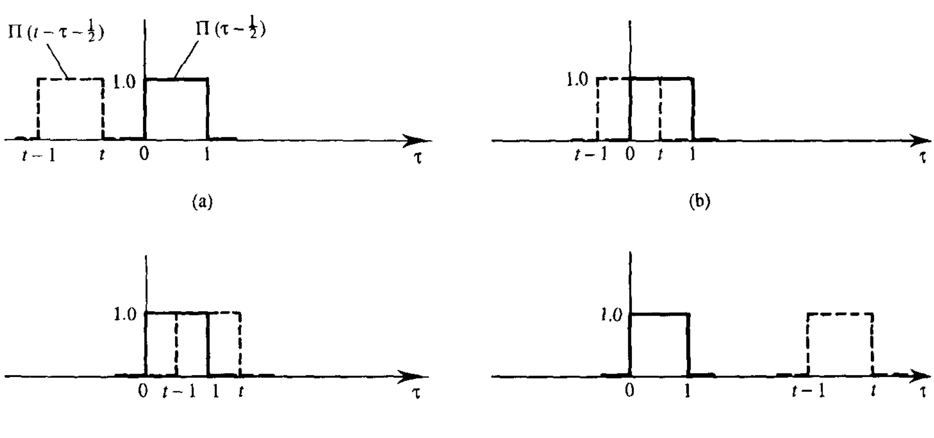


Figure 2.35 *Illustration (a) – (d) of self convolution of a rectangular pulse, Example 2.5.*

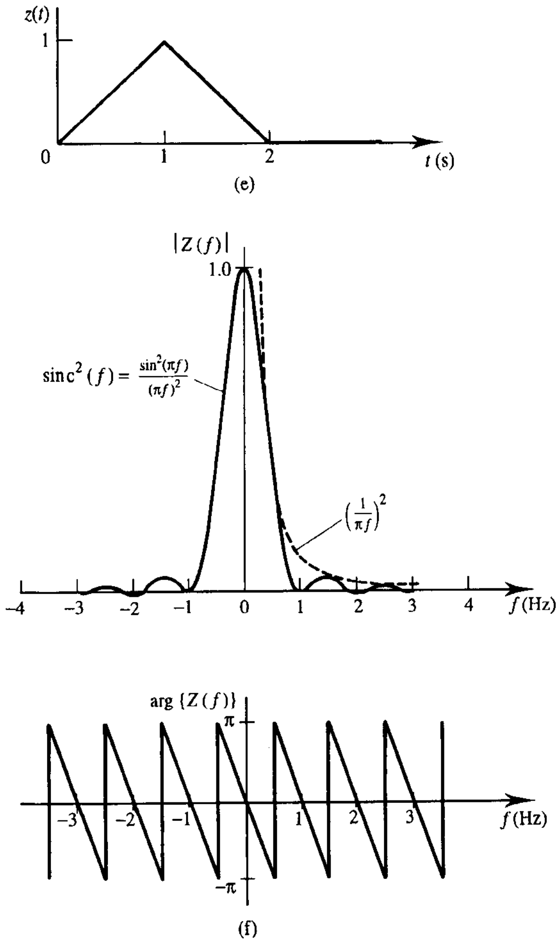


Figure 2.35-ctd. *Convolution result (e) and corresponding amplitude and phase spectra (f).*

$\Pi(t)$ is 1.0 whilst the width of $\Lambda(t)$ is 2.0.) Since the triangular function is centred here on $t = 1$ then we can use the time delay theorem and the tables of Fourier transform pairs to obtain:

$$\Lambda(t-1) \overset{FT}{\Leftrightarrow} \text{sinc}^2(f)\, e^{-j\omega 1}$$

The square of the Fourier transform of $\Pi(t - ½)$, using the time delay theorem and the table of pairs again, is given by:

$$\left[\text{FT}\left\{\Pi\left(t - \frac{1}{2}\right)\right\}\right]^2 = \left[\text{sinc}(f)\, e^{-j\omega/2}\right]^2$$

$$= \text{sinc}^2(f)\, e^{-j\omega}$$

The amplitude and phase spectra of $z(t)$ are given by:

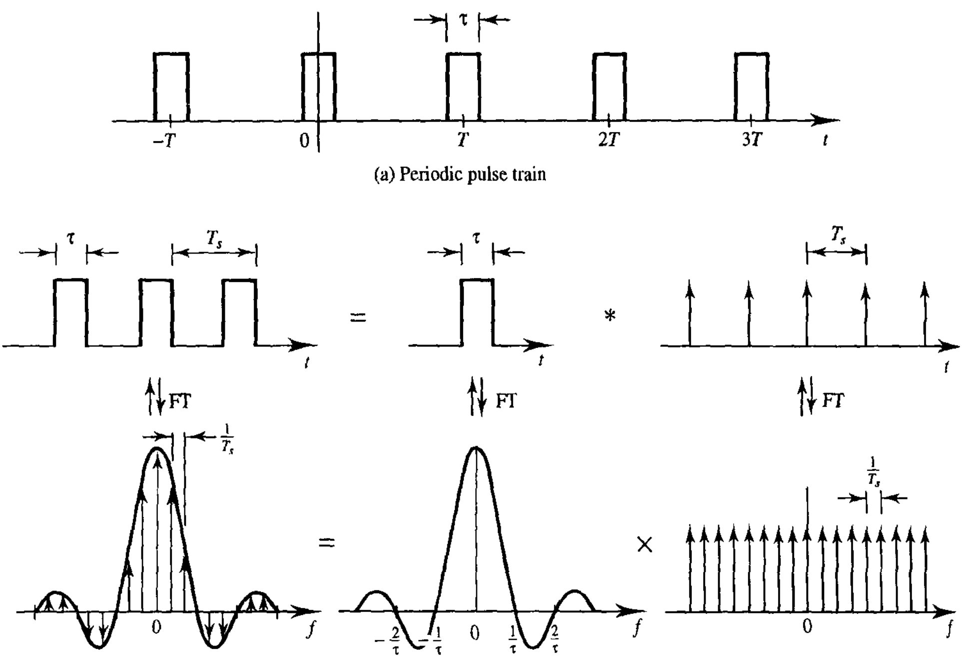
$$|Z(f)| = \text{sinc}^2(f) \quad \text{(V/Hz)}$$

$$\arg(Z(f)) = -2\pi f \quad \text{(rad)}$$

and are shown in Figure 2.35(f). (Notice that the phase spectrum really takes this form of a simple straight line with intercept zero, and gradient -2π (rad/Hz). It is conventional, however, to constrain its plot to the range $[-\pi, \pi]$ or sometimes $[0, 2\pi]$.)

Fourier transforms and Fourier series are, clearly, closely related. In fact by using the impulse function (section 2.3.3) a Fourier transform of a periodic function can be defined. Consider the periodic rectangular pulse stream $\Sigma_{n=-\infty}^{\infty}\Pi[(t-nT)/\tau]$, shown in Figure 2.36(a). This periodic waveform can be represented by the convolution of a transient signal (corresponding to the single period given by $n = 0$) with the periodic impulse train $\Sigma_{n=-\infty}^{\infty}\delta(t-nT)$, i.e.:

$$\sum_{n=-\infty}^{\infty} \Pi\left(\frac{t-nT}{\tau}\right) = \Pi\left(\frac{t}{\tau}\right) * \sum_{n=-\infty}^{\infty} \delta(t-nT) \qquad (2.51(a))$$



(b) Time and frequency domain representation of a periodic pulse train showing spectral lines arising from periodicity and spectral envelope arising from pulse shape

Figure 2.36 *Time and frequency domain representation of a periodic pulse train.*

(Each impulse in the impulse train reproduces the rectangular pulse in the convolution process.) In the same way that the Fourier transform of a single impulse can be defined (as a limiting case) the Fourier transform of an impulse train is defined (in the limit) as another impulse train. There is the usual relationship between the width (or period) of the impulse train in time and frequency domains, i.e.:

$$\sum_{k=-\infty}^{\infty} \delta(t - kT_s) \overset{\text{FT}}{\Leftrightarrow} f_s \sum_{n=-\infty}^{\infty} \delta(f - nf_s) \qquad (2.51(b))$$

where the time domain period, T_s, is the reciprocal of the frequency domain period, f_s. (The subscript s is used because the impulse train in communications engineering is often employed as a 'sampling function', T_s and f_s, in this context, being the sampling period and sampling frequency respectively.) The voltage spectrum of a rectangular pulse train can therefore be obtained by taking the Fourier transform of equation (2.51(a)), Figure 2.36(b), i.e.:

$$\text{FT}\left\{\sum_{n=-\infty}^{\infty} \Pi\left(\frac{t - nT}{\tau}\right)\right\} = \text{FT}\left\{\Pi\left(\frac{t}{\tau}\right)\right\}\text{FT}\left\{\sum_{n=-\infty}^{\infty} \delta(t - nT)\right\}$$

$$= \tau \operatorname{sinc}(\tau f)\frac{1}{T}\sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T}\right) \quad \text{(V/Hz)} \qquad (2.51(c))$$

This shows that the spectrum is given by a periodic impulse train (i.e. a line spectrum) with impulse (or line) separation of $1/T$ and impulse (or line) strength of $(\tau/T)\operatorname{sinc}(\tau f)$. ($(\tau/T)\operatorname{sinc}(\tau f)$ is usually called the spectrum envelope and, although real here, is potentially complex.)

The technique demonstrated here works for any periodic waveform, the separation of spectral lines being given by $1/T$ and the spectral envelope being given by $1/T$ times the Fourier transform of the single period contained in interval $[-T/2, T/2]$.

EXAMPLE 2.6

Sketch the Fourier spectra for a rectangular pulse train comprising pulses of amplitude A V and width 0.05 s with the following pulse repetition periods: (a) ¼ s; (b) ½ s; (c) 1 s.

The spectral envelope is controlled by the rectangular pulses of width 0.05 s. The spectrum is sinc x shaped, Figure 2.32, with the first zeros at $\pm 1/0.05 = \pm 20$ Hz. In all cases the waveform is periodic so the frequency spectrum can be represented by a Fourier series in which the lines, Figure 2.37, are spaced by $1/T_s$ Hz where T_s is the period in seconds.

Thus for (a) the lines occur every 4 Hz and the 0 Hz component, C_0 in equation (2.15), has a magnitude of $A/20 \times 1/T_s = 4A/20 = A/5$. The other components $C_1, C_2, C_3, \cdots, C_n$ follow the sinc x envelope as shown in Figure 2.37(b).

For the case (b) where T_s is ½ s then the spectral lines are now $1/T_s = 2$ Hz apart which is half the spacing of the ¼ s period case in part (a). The envelope of the Fourier spectrum is unaltered but the C_0 term reduces in amplitude owing to the longer period. Thus $C_0 = A/20 \times 2 = A/10$ and the waveform and spectrum are shown in Figure 2.37(b).

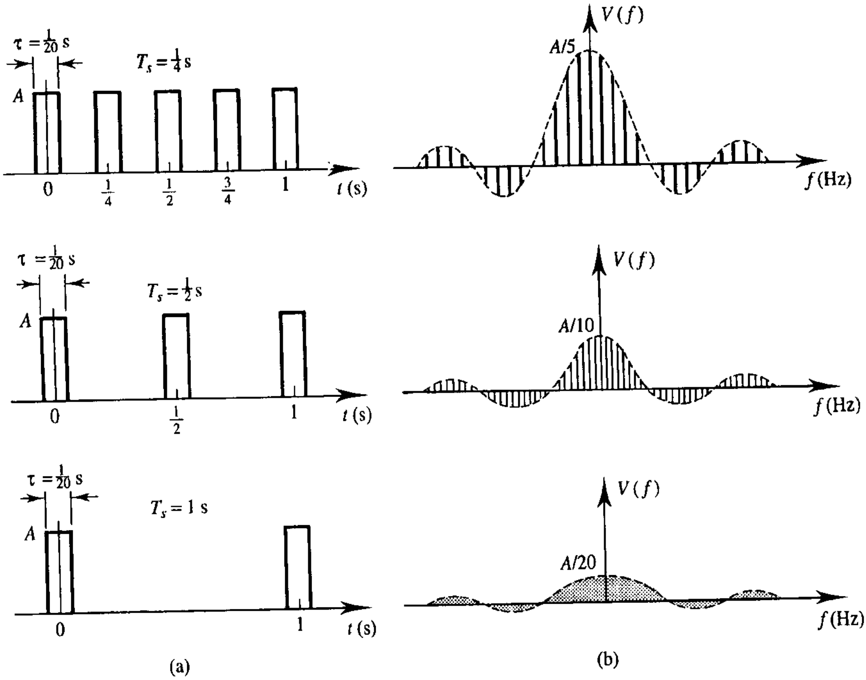


Figure 2.37 *Pulsed waveform and corresponding frequency spectra with specific values for pulse repetition period.*

For (c) the line spacing becomes 1 Hz and the amplitude at 0 Hz drops to $A/20$.

2.4 Power and energy spectra

As an alternative to plotting peak or RMS voltage against frequency the quantity:

$$G_1(f) = |V_{RMS}(f)|^2 \ (\text{V}^2) \tag{2.52(a)}$$

can be plotted for periodic signals. This is a line spectrum the ordinate of which has units of V^2 (or watts in a 1 Ω resistive load) representing *normalised* power. (The subscript 1 indicates that the spectrum is single sided.) If the impedance level, R, is not 1 Ω then the absolute (i.e. non-normalised) power spectrum is given by:

$$G_1(f) = \frac{|V_{RMS}(f)|^2}{R} \ (\text{W}) \tag{2.52(b)}$$

Figure 2.38(a) shows such a power spectrum for a periodic signal. Although each line in Figure 2.38(a) no longer represents a rotating phasor, two sided power spectra are still often defined by associating half the power in each spectral line with a negative frequency, Figure 2.38(b). Notice that this means that the total power in a signal is the sum of the powers in all its spectral lines irrespective of whether a one or two sided spectral representation is being used.

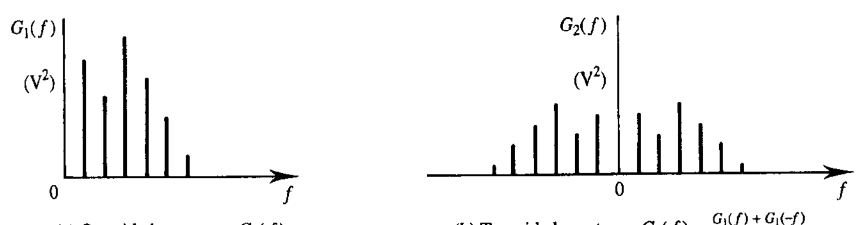
For a transient signal the two sided voltage spectrum $V(f)$ has units of V/Hz and the quantity:

$$E_2(f) = |V(f)|^2 \quad (V^2\ s/Hz) \tag{2.53(a)}$$

therefore has units of V^2/Hz^2 or V^2 s/Hz. The corresponding non-normalised spectrum is given by:

$$E_2(f) = \frac{|V(f)|^2}{R} \quad (J/Hz) \tag{2.53(b)}$$

where R is load resistance (in Ω). The quantity $E_2(f)$ now has units of W s/Hz or J/Hz and is therefore called an *energy* spectral density. Like power spectra, energy spectra can be presented as two or one sided, Figure 2.39(a) and (b). Note that energy spectra are always continuous (never discrete) whilst power spectra can be either discrete (as is the case for periodic waveforms) or continuous (as is the case for random signals). Continuous power spectra, i.e. power spectral densities, are discussed in section 3.3.3 and, in the context of linear systems, in section 4.6.1.



(a) One sided spectrum, $G_1(f)$ (b) Two sided spectrum, $G_2(f) = \frac{G_1(f) + G_1(-f)}{2}$

Figure 2.38 *Power spectra of a periodic signal.*

(a) One sided spectrum, $E_1(f)$ (b) Two sided spectrum, $E_2(f) = \frac{E_1(f) + E_1(-f)}{2}$

Figure 2.39 *Energy spectral densities of a transient signal.*

2.5 Generalised orthogonal function expansions

Fourier series and transforms constitute a special case of a more general mathematical technique, namely the orthogonal function expansion. The concept of orthogonal functions is closely connected with that of orthogonal (i.e. perpendicular) vectors. For this reason the important characteristics and properties of vectors are now reviewed.

2.5.1 Review of vectors

Vectors possess magnitude, direction and sense. They can be added, Figure 2.40(a), i.e.:

$$\mathbf{a}, \mathbf{b} \rightarrow \mathbf{a} + \mathbf{b}$$

and multiplied by a scalar, λ, Figure 2.40(b), i.e.:

$$\mathbf{a} \rightarrow \lambda \mathbf{a}$$

Their properties [Spiegel] include commutation, distribution and association, i.e.:

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a} \tag{2.54(a)}$$

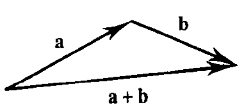
$$\lambda(\mathbf{a} + \mathbf{b}) = \lambda \mathbf{a} + \lambda \mathbf{b} \tag{2.54(b)}$$

$$\lambda(\mu \mathbf{a}) = (\lambda \mu)\mathbf{a} \tag{2.54(c)}$$

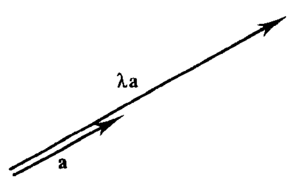
A scalar product of two vectors, Figure 2.41, can be defined by:

$$\mathbf{a} \,.\, \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta \tag{2.55}$$

where θ is the angle between the vectors and the modulus $|\;|$ indicates their length or magnitude.



(a) Addition



(b) Multiplication by a scalar

Figure 2.40 *Fundamental vector operations.*

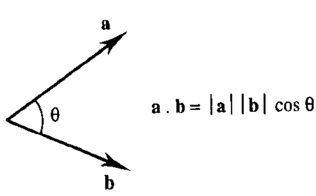


Figure 2.41 *Scalar product of vectors.*

The distance between two vectors, i.e. their difference, is found by reversing the sense of the vector to be subtracted and adding, Figure 2.42:

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}) \tag{2.56}$$

A vector in three dimensions can be specified as a weighted sum of any three non-coplanar *basis* vectors:

$$\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 \tag{2.57}$$

If the vectors are mutually perpendicular, i.e.:

$$\mathbf{e}_1 \cdot \mathbf{e}_2 = \mathbf{e}_2 \cdot \mathbf{e}_3 = \mathbf{e}_3 \cdot \mathbf{e}_1 = 0 \tag{2.58(a)}$$

then the three vectors are said to form an *orthogonal* set. If, in addition, the three vectors have unit length:

$$\mathbf{e}_1 \cdot \mathbf{e}_1 = \mathbf{e}_2 \cdot \mathbf{e}_2 = \mathbf{e}_3 \cdot \mathbf{e}_3 = 1 \tag{2.58(b)}$$

then they are said to form an *orthonormal* set. The above concepts can be extended to vectors with any number of dimensions, N. Such a vector can be represented as the sum of N basis vectors, i.e.:

$$\mathbf{x} = \sum_{i=1}^{N} \lambda_i \mathbf{e}_i \tag{2.59}$$

If $\mathbf{e}_i$ is an orthonormal basis it is easy to find the values of λ_i:

$$\mathbf{x} \cdot \mathbf{e}_j = \sum_{i=1}^{N} \lambda_i \mathbf{e}_i \cdot \mathbf{e}_j = \lambda_j \tag{2.60}$$

(since $\mathbf{e}_i \cdot \mathbf{e}_j = 0$ for $i \neq j$ and $\mathbf{e}_i \cdot \mathbf{e}_j = 1$ for $i = j$). Scalar products of vectors expressed using orthonormal bases are also simple to calculate:

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{N} \lambda_i \mathbf{e}_i \cdot \sum_{j=1}^{N} \mu_j \mathbf{e}_j \tag{2.61(a)}$$

i.e.:

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{N} \lambda_i \mu_i \tag{2.61(b)}$$

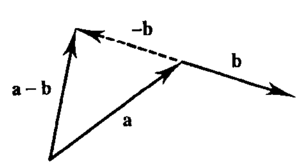


Figure 2.42 *Subtraction of vectors.*

Equation (2.61(b)) is the most general form of Parseval's theorem, equation (2.28). A special case of this theorem is:

$$\mathbf{x}\,.\,\mathbf{x} = \sum_{i=1}^{N} \lambda_i^2 \tag{2.61(c)}$$

which has already been discussed in the context of the power contained in a periodic waveform (section 2.2.3).

If the number of basis vectors available to express an N-dimensional vector is limited to M ($M < N$) then, provided the basis is orthonormal, the best approximation (in a least square error sense) is given by:

$$\mathbf{x} \simeq \sum_{i=1}^{M} \lambda_i \mathbf{e}_i = \mathbf{x}_M \tag{2.62}$$

where $\lambda_i = \mathbf{x}\,.\,\mathbf{e}_i$ as before. This is easily proved by calculating the squared error:

$$\begin{aligned} |\mathbf{x} - \mathbf{x}_M|^2 &= \left| \mathbf{x} - \sum_{i=1}^{M} \lambda_i \mathbf{e}_i \right|^2 \\ &= |\mathbf{x}|^2 - 2\sum_{i=1}^{M} \lambda_i (\mathbf{e}_i\,.\,\mathbf{x}) + \sum_{i=1}^{M} \lambda_i^2 \\ &= |\mathbf{x}|^2 + \sum_{i=1}^{M} (\lambda_i - \mathbf{e}_i\,.\,\mathbf{x})^2 - \sum_{i=1}^{M} (\mathbf{e}_i\,.\,\mathbf{x})^2 \end{aligned} \tag{2.63}$$

The right hand side of equation (2.63) is clearly minimised by putting $\lambda_i = \mathbf{x}\,.\,\mathbf{e}_i$.

From the definition of the scalar product it is apparent that:

$$|\mathbf{x}\,.\,\mathbf{y}| \leq |\mathbf{x}|\,|\mathbf{y}| \tag{2.64}$$

This holds for any kind of vector providing that the scalar product is defined to satisfy:

$$\mathbf{x}\,.\,\mathbf{x} \geq 0, \text{ for all } \mathbf{x} \tag{2.65(a)}$$

$$\mathbf{x}\,.\,\mathbf{x} = 0, \text{ only if } \mathbf{x} = \mathbf{0} \tag{2.65(b)}$$

where $\mathbf{0}$ is a null vector. (Equation (2.64) is a particularly simple form of the Schwartz inequality, see equation 2.71(a).) In order to satisfy equation (2.65(a)) the scalar product of *complex* vectors must be defined by:

$$\mathbf{x}\,.\,\mathbf{y} = \sum_{i=1}^{N} \lambda_i^* \mu_i \tag{2.66}$$

2.5.2 Vector interpretation of waveforms

Nyquist's sampling theorem (section 2.2.5) asserts that a *periodic* signal having a highest frequency component located at f_H Hz is *fully* specified by N samples spaced $1/2f_H$ s

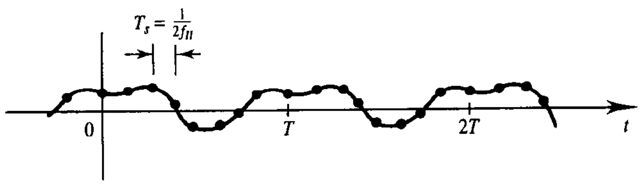


Figure 2.43 *Nyquist sampling of a periodic function (N = 8).*

apart, Figure 2.43. Since all the 'information' in a periodic waveform is contained in this set of independent samples (which repeat indefinitely), each sample value can be regarded as being the length of one vector belonging to an orthogonal basis set. A waveform requiring N samples per period for its specification can therefore be interpreted as an N-dimensional vector. For sampling at the Nyquist rate the intersample spacing is $T_s = 1/2f_H$ and:

$$N = T/T_s = 2f_H T \tag{2.67(a)}$$

where T is the period of the waveform. In a slightly more general form:

$$N = 2BT \tag{2.67(b)}$$

where B is waveform bandwidth. This is called the dimensionality theorem. For transient signals an infinite number of samples would be required to retain all the signal's information.

Functions can be added and scaled in the same way as vectors to produce new functions. A scalar product for certain periodic signals can be defined as a continuous version of equation (2.66) but with a factor $1/T'$ so that the scalar product has dimensions of V^2 and can be interpreted as a *cross-power*, i.e.:

$$[f(t), g(t)] = \frac{1}{T'}\int_0^{T'} f^*(t)\, g(t)\, dt$$

$$= \langle f^*(t)\, g(t)\rangle \ (V^2) \tag{2.68(a)}$$

where T' is the period of the product $f^*(t)g(t)$. (The notation $[f(t), g(t)]$ is used here to denote the scalar product of functions rather than $\mathbf{f} \,.\, \mathbf{g}$ as used for vectors.) More generally, the definition adopted is:

$$[f(t), g(t)] = \lim_{T'\to\infty} \frac{1}{T'}\int_{-T'/2}^{T'/2} f^*(t)\, g(t)\, dt$$

$$= \langle f^*(t)\, g(t)\rangle \ (V^2) \tag{2.68(b)}$$

since this includes the possibility that $f^*(t)g(t)$ has infinite period. The corresponding definition for transient signals is a *cross-energy*, i.e.:

$$[f(t), g(t)] = \int_{-\infty}^{\infty} f^*(t)\, g(t)\, dt \quad (V^2\ s) \tag{2.68(c)}$$

The scalar products of periodic and transient signals with themselves, which represent average signal power and total signal energy respectively, are therefore:

$$[f(t), f(t)] \equiv \frac{1}{T} \int_0^T |f(t)|^2 \, dt \quad (\mathrm{V}^2) \tag{2.69(a)}$$

$$[f(t), f(t)] \equiv \int_{-\infty}^{\infty} |f(t)|^2 \, dt \quad (\mathrm{V}^2 \ \mathrm{s}) \tag{2.69(b)}$$

Interpreting the signals as vectors, equations (2.69) correspond to finding the vector's square magnitude, i.e.:

$$\mathbf{x} \,.\, \mathbf{x} = \sum_{i=1}^{N} \lambda_i^* \lambda_i$$

$$= \sum_{i=1}^{N} |\lambda_i|^2 = |\mathbf{x}|^2 \tag{2.70}$$

Using the definition of a scalar product for complex vectors, equation (2.66), the Schwartz inequality, equation (2.64), becomes:

$$\left| \int_{-\infty}^{\infty} f^*(t)\, g(t)\, dt \right| \leq \left[\int_{-\infty}^{\infty} |f(t)|^2 \, dt \right]^{1/2} \left[\int_{-\infty}^{\infty} |g(t)|^2 \, dt \right]^{1/2} \tag{2.71(a)}$$

where the equality holds if and only if:

$$g(t) = C\, f^*(t) \tag{2.71(b)}$$

Equations (2.71(a)) and (b) can be used to derive the optimum response of predetection filters in digital communications receivers.

2.5.3 Orthogonal and orthonormal signals

Consider a periodic signal, $f(t)$, with only three dimensions[1], i.e. with $N = 2BT = 3$, Figure 2.44. In sample space, Figure 2.45(a), the signal is represented by:

$$\mathbf{f} = \sum_{i=1}^{3} F_i \, \hat{\mathbf{a}}_i \tag{2.72(a)}$$

where $\hat{\mathbf{a}}_i$ represents an orthonormal sample set. The same function **f** could, however, be described in a second orthonormal coordinate system, Figure 2.45(b), rotated with respect

[1] In reality the number of dimensions, N, must be even since for a periodic signal the maximum frequency, B Hz, must be an integer multiple, n, of the fundamental frequency $1/T$ Hz. The dimensionality theorem can therefore be written as $N = 2(n/T)T = 2n$. Choosing $N = 2$, however, trivialises this in that the orthogonal functions become phasors whilst choosing $N = 4$ precludes signal vector visualisation in 3-dimensional space.

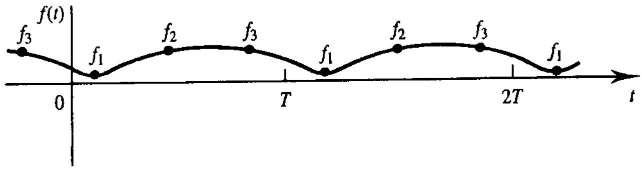


Figure 2.44 *Three dimensional (i.e. 3-sample) function.*

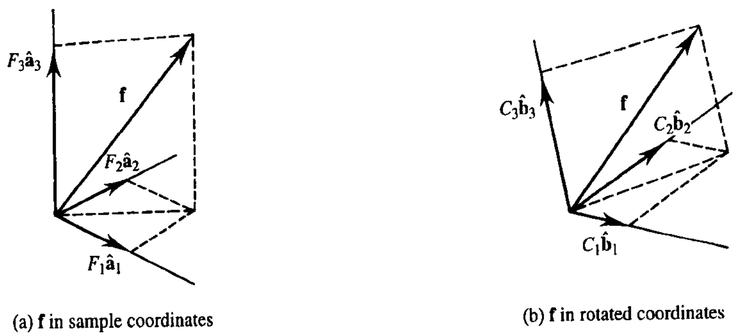


Figure 2.45 *Vector interpretation of a 3-dimensional signal.*

to the first, i.e.:

$$\mathbf{f} = \sum_{i=1}^{3} C_i \, \hat{\mathbf{b}}_i \tag{2.72(b)}$$

Each unit vector $\hat{\mathbf{b}}_j$ itself represents a three sample function:

$$\hat{\mathbf{b}}_j = \sum_{i=1}^{3} (\hat{\mathbf{b}}_j \cdot \hat{\mathbf{a}}_i)\hat{\mathbf{a}}_i \tag{2.73}$$

This demonstrates the important idea that a function or signal having N dimensions (in this case three) can be expressed in terms of a weighted sum of N other orthonormal, N-dimensional, functions. (Actually these *basis* functions do not have to be orthonormal or even orthogonal providing none can be exactly expressed as a linear sum of the others.)

Generalising the vector notation to make it more appropriate for signals (which may include the case of transient signals where $N = \infty$) $\mathbf{f}$ is replaced by $f(t)$ and $\mathbf{b}_i$ (which represents an orthogonal but not necessarily orthonormal set) is replaced by $\phi_i(t)$. Equation (2.72(b)) then becomes:

$$f(t) = \sum_{i=1}^{N} C_i \, \phi_i(t) \tag{2.74}$$

If the basis function set $\phi_i(t)$ is orthonormal over an interval $[a, b]$ then:

$$\int_a^b \phi_i(t)\,\phi_j^*(t)\,dt = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \tag{2.75}$$

which corresponds to the vector property:

$$\mathbf{b}_i \,.\, \mathbf{b}_j = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \tag{2.76}$$

If the functions are orthogonal but not orthonormal then the upper expression on the right hand side of equation (2.75) does not apply.

EXAMPLE 2.7
Consider the functions shown in Figure 2.46. Do these functions form an orthogonal set over the range [−1, 1]? Do these functions form an orthonormal set over the same range?

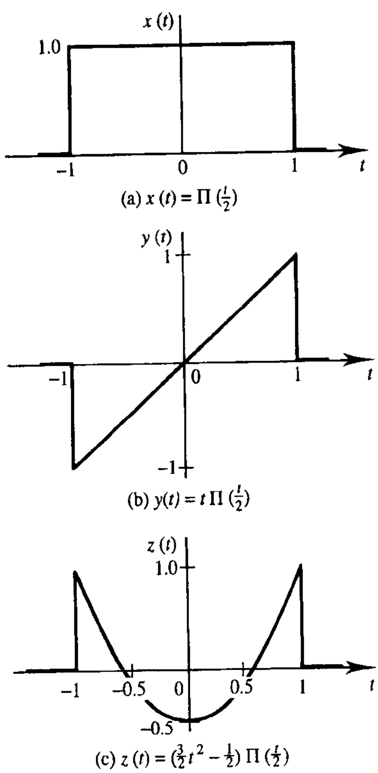


Figure 2.46 *Three functions tested for orthogonality and orthonormality in Example 2.7.*

To determine whether the functions form an orthogonal set we use equation (2.75):

$$\int_a^b \phi_i(t)\,\phi_j^*(t)\,dt = 0 \quad (i \neq j)$$

(Since all the functions are real the conjugate symbol is immaterial here.)

First examine $x(t)$ and $y(t)$. Since the product is odd about zero the integral must be zero, i.e. $x(t)$ and $y(t)$ are orthogonal.

Now examine $x(t)$ and $z(t)$.

$$\int_{-1}^{1} x(t)\,z(t)\,dt = \int_{-1}^{1} \left(\frac{3}{2}t^2 - \frac{1}{2}\right) dt$$

$$= \frac{3}{2}\left[\frac{t^3}{3}\right]_{-1}^{1} - \frac{1}{2}\left[\,t\,\right]_{-1}^{1} = 0$$

i.e. $x(t)$ and $z(t)$ are orthogonal.

Finally we examine $y(t)$ and $z(t)$. Here the product is again odd about zero and the integral is therefore zero by inspection, i.e. $y(t)$ and $z(t)$ are orthogonal.

To establish the normality or otherwise of the functions we test the square integral against 1.0:

$$\int_a^b |\,\phi_i(t)\,|^2\,dt = 1 \quad \text{for normal } \phi_i(t)$$

The square integral of $x(t)$ is 2.0 by inspection. We need go no further, therefore, since if any function in the set fails this test then the set is not orthonormal.

2.5.4 Evaluation of basis function coefficients

Equation (2.74) can be multiplied by $\phi_j^*(t)$ to give:

$$f(t)\,\phi_j^*(t) = \sum_{i=1}^{N} C_i\,\phi_i(t)\,\phi_j^*(t) \tag{2.77}$$

Integrating and reversing the order of integration and summation on the right hand side:

$$\int_a^b f(t)\,\phi_j^*(t)\,dt = \sum_{i=1}^{N} C_i \int_a^b \phi_i(t)\,\phi_j^*(t)\,dt \tag{2.78}$$

Since the integral on the right hand side is zero for all $i \neq j$ equation (2.78) can be rewritten as:

$$\int_a^b f(t)\,\phi_j^*(t)\,dt = C_j \int_a^b |\phi_j(t)|^2\,dt \tag{2.79}$$

and rearranging equation (2.79) gives an explicit formula for C_j, i.e.:

$$C_j = \frac{\int_a^b f(t)\, \phi_j^*(t)\, dt}{\int_a^b |\phi_j(t)|^2\, dt} \tag{2.80}$$

If the basis functions $\phi_i(t)$ are orthonormal over the range $[a, b]$ then this reduces to:

$$C_j = \int_a^b f(t)\, \phi_j^*(t)\, dt \tag{2.81}$$

For the special case of $\phi_j(t) = e^{j2\pi f_j t}$, $a = t$ and $b = t + T$, equation (2.81) gives the coefficients for an orthogonal function expansion in terms of a set of cisoids. This, of course, is identical to $T\tilde{C}'_n$ in the Fourier series of equation (2.25).

2.5.5 Error energy and completeness

When a function is *approximated* by a superposition of N basis functions over some range, T, i.e.:

$$f(t) \simeq f_N(t) \tag{2.82(a)}$$

where:

$$f_N(t) = \sum_{i=1}^{N} C_i\, \phi_i(t) \tag{2.82(b)}$$

then the 'error energy', E_e, is given by:

$$E_e = \int_t^{t+T} |f(t) - f_N(t)|^2\, dt \tag{2.83}$$

(Note that E_e/T is the mean square error.) The basis set $\phi_i(t)$ is said to be *complete* over the interval T, for a given class of signals, if $E_e \to 0$ as $N \to \infty$ for those signals. Calculation of the coefficients in an orthogonal function expansion using equation (2.80) or (2.81) results in a minimum error energy approximation.

EXAMPLE 2.8

The functions shown in Figure 2.47(a) are the first four elements of the orthonormal set of Walsh functions [Beauchamp, Harmuth]. Using these as basis functions find a minimum error energy approximation for the function, $f(t)$, shown in Figure 2.47(b). Sketch the approximation.

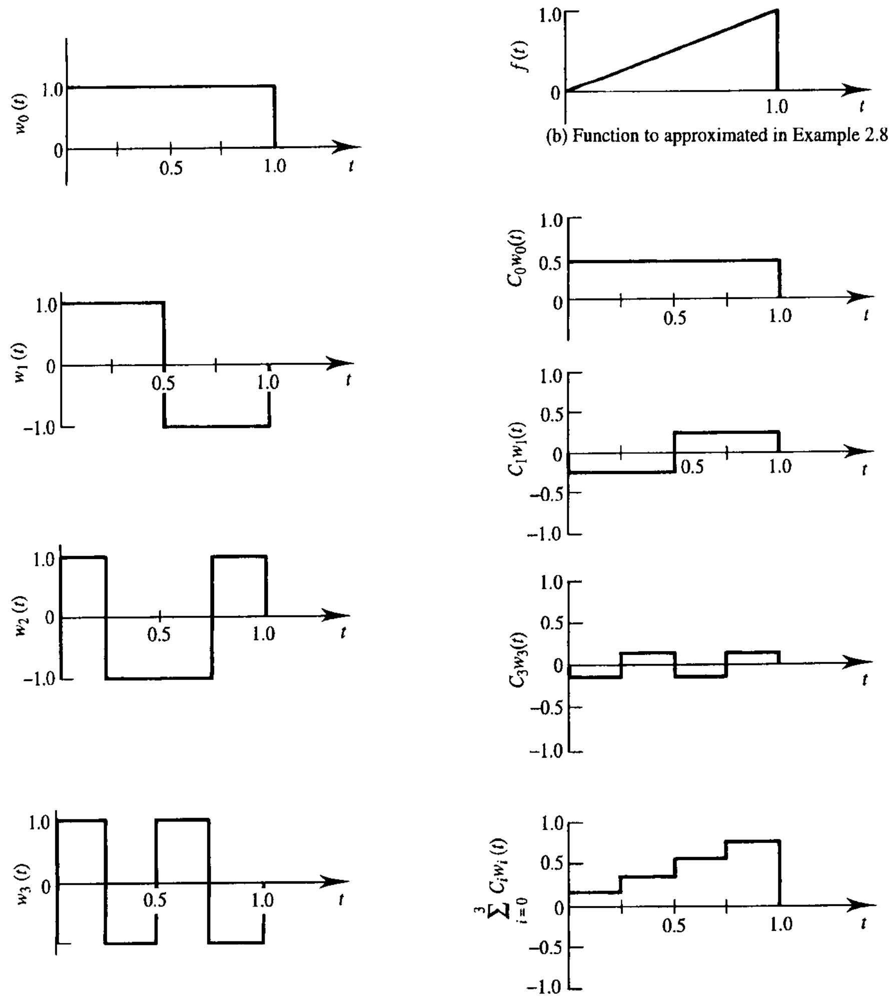


Figure 2.47 *Orthonormal function expansion using Walsh functions.*

Each of the coefficients is found in turn using equation (2.81), i.e.:

$$C_0 = \int_0^1 f(t)\, w_0(t)\, dt = \int_0^1 t\, dt = \left[\frac{t^2}{2}\right]_0^1 = 0.5$$

$$C_1 = \int_0^1 f(t)\, w_1(t)\, dt = \int_0^{0.5} t\, dt + \int_{0.5}^{1.0} -t\, dt$$

$$= \left[\frac{t^2}{2}\right]_0^{0.5} - \left[\frac{t^2}{2}\right]_{0.5}^{1.0} = -0.25$$

$$C_2 = \int_0^1 f(t)\, w_2(t)\, dt = \int_0^{0.25} t\, dt + \int_{0.25}^{0.75} -t\, dt + \int_{0.75}^{1.0} t\, dt$$

$$= \left[\frac{t^2}{2}\right]_0^{0.25} - \left[\frac{t^2}{2}\right]_{0.25}^{0.75} + \left[\frac{t^2}{2}\right]_{0.75}^{1.0} = 0$$

$$C_3 = \int_0^1 f(t)\, w_3(t)\, dt = \int_0^{0.25} t\, dt + \int_{0.25}^{0.5} -t\, dt + \int_{0.5}^{0.75} t\, dt + \int_{0.75}^{1.0} -t\, dt$$

$$= \left[\frac{t^2}{2}\right]_0^{0.25} - \left[\frac{t^2}{2}\right]_{0.25}^{0.5} + \left[\frac{t^2}{2}\right]_{0.5}^{0.75} - \left[\frac{t^2}{2}\right]_{0.75}^{1.0} = -0.125$$

The minimum error energy approximation is therefore given in equation (2.82(b)) by:

$$f_N(t) = \sum_{i=0}^{3} C_i\, w_i(t)$$

$$= 0.5w_0(t) - 0.25w_1(t) - 0.125w_3(t)$$

The approximation $f_N(t)$ is sketched in Figure 2.47(c).

2.6 Correlation functions

Attention is restricted here to *real* functions and signals. The scalar product of two transient signals, $v(t)$ and $w(t)$, defined by equation (2.68(c)), and repeated here for convenience, is therefore:

$$[v(t), w(t)] = \int_{-\infty}^{\infty} v(t)\, w(t)\, dt \qquad (2.84(a))$$

Since this quantity is a measure of similarity between the two signals it is usually called the *(cross) correlation* of $v(t)$ and $w(t)$, normally denoted by $R_{vw}(0)$, i.e.:

$$R_{vw}(0) = [v(t),\, w(t)] \qquad (2.84(b))$$

(Recall that $[v(t),\, w(t)]$ in section 2.5.2 was called a cross-energy.) More generally a cross correlation *function*, $R_{vw}(\tau)$, can be defined, i.e.:

$$R_{vw}(\tau) = [v(t),\, w(t-\tau)]$$

$$= \int_{-\infty}^{\infty} v(t)\, w(t-\tau)\, dt \qquad (2.85)$$

This is a measure of the similarity between $v(t)$ and a time shifted version of $w(t)$. The value of $R_{vw}(\tau)$ depends not only on the similarity of the signals, however, but also on their magnitude. This magnitude dependence can be removed by normalising both functions such that their associated normalised energies are unity, i.e.:

$$\rho_{vw}(\tau) = \frac{\int_{-\infty}^{\infty} v(t)\, w(t-\tau)\, dt}{\sqrt{\left(\int_{-\infty}^{\infty} |v(t)|^2\, dt\right)}\sqrt{\left(\int_{-\infty}^{\infty} |w(t)|^2\, dt\right)}} \tag{2.86}$$

The normalised cross correlation function, $\rho_{vw}(\tau)$, has the following properties:

1. $-1 \leq \rho_{vw}(\tau) \leq 1$
2. $\rho_{vw}(\tau) = -1$ if, and only if, $v(t) = -kw(t-\tau)$
3. $\rho_{vw}(\tau) = 1$ if, and only if, $v(t) = kw(t-\tau)$

$\rho_{vw}(\tau) = 0$ indicates that $v(t)$ and $w(t-\tau)$ are orthogonal and hence have no similarity whatsoever. (Later, in Chapter 11, we will use two separate parallel channel carriers to construct a four-phase modulator. By using orthogonal carriers $\sin \omega t$ and $\cos \omega t$ we ensure that there is no interference between the parallel channels, equation (2.21(c)).)

For real *periodic* waveforms, $p(t)$ and $q(t)$, the correlation or scalar product, defined in equation (2.68(b)) and repeated here, is:

$$\begin{aligned} R_{pq}(0) &= [p(t), q(t)] \\ &= \lim_{T' \to \infty} \frac{1}{T'} \int_{-T'/2}^{T'/2} p(t)\, q(t)\, dt \\ &= \langle p(t)\, q(t) \rangle \end{aligned} \tag{2.87}$$

The generalisation to a cross correlation function is therefore:

$$\begin{aligned} R_{pq}(\tau) &= [p(t), q(t-\tau)] \\ &= \lim_{T' \to \infty} \frac{1}{T'} \int_{-T'/2}^{T'/2} p(t)\, q(t-\tau)\, dt \end{aligned} \tag{2.88}$$

and the normalised cross correlation function, $\rho_{pq}(\tau)$, becomes:

$$\begin{aligned} \rho_{pq}(\tau) &= \frac{\langle p(t)\, q(t-\tau) \rangle}{\sqrt{\langle |p(t)|^2 \rangle}\ \sqrt{\langle |q(t)|^2 \rangle}} \\ &= \frac{\lim_{T' \to \infty} \left(\frac{1}{T'}\right) \int_{-T'/2}^{T'/2} p(t)\, q(t-\tau)\, dt}{\sqrt{\left(\left(\frac{1}{T}\right)\int_{t}^{t+T} |p(t)|^2\, dt\right)}\ \sqrt{\left(\left(\frac{1}{T}\right)\int_{t}^{t+T} |q(t)|^2\, dt\right)}} \end{aligned} \tag{2.89}$$

(Notice that the denominator of equation (2.89) is the geometric mean of the normalised powers of $p(t)$ and $q(t)$. For periodic signals $\rho_{pq}(\tau)$ therefore represents $R_{pq}(\tau)$ after $p(t)$ and $q(t)$ have been normalised to an RMS value of 1.0.)

EXAMPLE 2.9

Find the normalised cross correlation function of the sinusoid and the square wave shown in Figure 2.48(a).

Since the periods of the two functions in this example are the same we can perform the averaging in equation (2.89) over the (finite) period of the product, T', i.e. :

$$\rho_{pq}(\tau) = \frac{\left(\frac{1}{T'}\right)\int_{-T'/2}^{T'/2} p(t)q(t-\tau)\,dt}{\sqrt{\left(\left(\frac{1}{T}\right)\int_0^T p^2(t)\,dt\right)}\sqrt{\left(\left(\frac{1}{T}\right)\int_0^T q^2(t)\,dt\right)}}$$

It is clear that here T' can be taken equal to the period, T, of the sinusoid and square wave. Furthermore the RMS values of the two functions are (by inspection) $1/\sqrt{2}$ and 1.0. The cross correlation function is therefore given by:

$$\rho_{pq}(\tau) = \frac{\frac{1}{0.02}\int_{-0.01}^{0.01}\left[\Pi\left(\frac{t-0.005}{0.01}\right)-\Pi\left(\frac{t+0.005}{0.01}\right)\right]\cos[2\pi\, 50(t-\tau)]\,dt}{(1/\sqrt{2})\times 1.0}$$

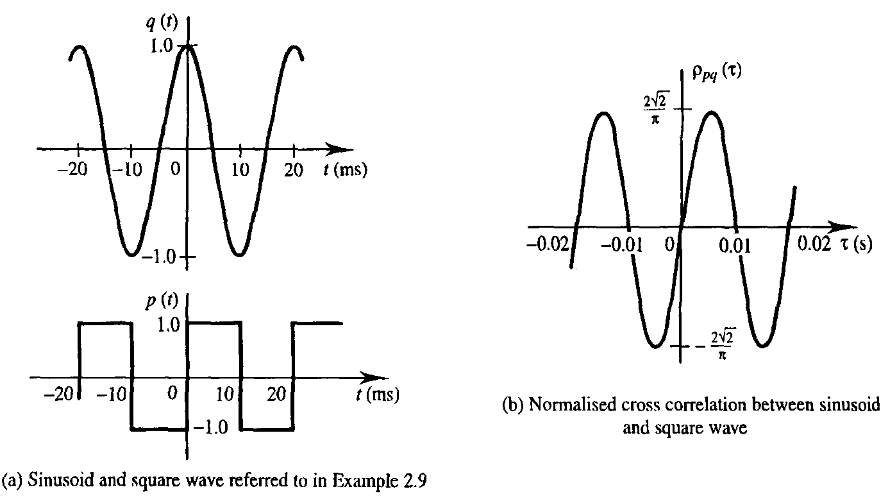


Figure 2.48 *Normalised cross correlation of sinusoid and square wave.*

$$= \frac{\sqrt{2}}{0.02}\left\{\int_0^{0.01} \cos[2\pi\, 50(t-\tau)]\, dt - \int_{-0.01}^{0} \cos[2\pi\, 50(t-\tau)]\, dt\right\}$$

Using change of variable $x = t - \tau$:

$$\rho_{pq}(\tau) = \frac{\sqrt{2}}{0.02}\left\{\int_{-\tau}^{0.01-\tau} \cos(2\pi\, 50x)\, dx - \int_{-\tau-0.01}^{-\tau} \cos(2\pi\, 50x)\, dx\right\}$$

$$= \frac{\sqrt{2}}{0.02}\left\{\left[\frac{\sin(2\pi\, 50x)}{2\pi\, 50}\right]_{-\tau}^{0.01-\tau} - \left[\frac{\sin(2\pi\, 50x)}{2\pi\, 50}\right]_{-\tau-0.01}^{-\tau}\right\}$$

$$= \frac{\sqrt{2}}{0.02\; 2\pi 50}\{\sin[2\pi 50(0.01-\tau)] + 2\sin(2\pi 50\tau) - \sin[2\pi 50(\tau + 0.01)]\}$$

$$= \frac{1}{\sqrt{2}\,\pi}\, 4\sin(2\pi\, 50\tau) = \frac{4}{\sqrt{2}\,\pi}\sin(2\pi\, 50\tau)$$

Figure 2.48(b) shows a sketch of $\rho_{pq}(\tau)$.

If the two signals being correlated are identical then the result is called the *auto*correlation function, $R_{vv}(\tau)$ or $R_v(\tau)$. For real transient signals the autocorrelation function is therefore defined by:

$$R_v(\tau) = \int_{-\infty}^{\infty} v(t)\, v(t-\tau)\, dt \tag{2.90(a)}$$

and for real periodic signals by:

$$R_p(\tau) = \frac{1}{T}\int_t^{t+T} p(t)\, p(t-\tau)\, dt \tag{2.90(b)}$$

Normalised autocorrelation functions can be defined by dividing equations (2.90(a)) and (b) by the energy in $v(t)$ and power in $p(t)$, respectively, dissipated in 1 Ω. There are several properties of (real signal) autocorrelation functions to note:

1. $R_x(\tau)$ is real.
2. $R_x(\tau)$ has even symmetry about $\tau = 0$, i.e.:

$$R_x(\tau) = R_x(-\tau) \tag{2.91}$$

3. $R_x(\tau)$ has a maximum (positive) magnitude at $\tau = 0$, i.e.:

$$|R_x(\tau)| \le R_x(0), \quad \text{for } any\ \tau \ne 0. \tag{2.92}$$

4a. If $x(t)$ is periodic and has units of V then $R_x(\tau)$ is also periodic (with the same period as $x(t)$) and has units of V^2 (i.e. normalised power).

4b. If $x(t)$ is transient and has units of V then $R_x(\tau)$ is also transient with units of V^2 s (i.e. normalised energy).

5a. The autocorrelation function of a transient signal and its (two sided) energy spectral density are a Fourier transform pair, i.e.:

$$R_v(\tau) \overset{\text{FT}}{\Leftrightarrow} E_v(f) \tag{2.93}$$

This theorem is proved as follows:

$$E_v(f) = |V(f)|^2 = V(f)\, V^*(f)$$

$$= \text{FT}\left\{ v(t) * v^*(-t) \right\} = \text{FT}\left\{ \int_{-\infty}^{\infty} v(t)\, v^*(-\tau + t)\, dt \right\}$$

$$= \text{FT}\,\{R_v(\tau)\} \quad (\text{V}^2\ \text{s/Hz}) \tag{2.94(a)}$$

i.e.:

$$E_v(f) = \int_{-\infty}^{\infty} R_v(\tau)\, e^{-j2\pi f\tau}\, d\tau \tag{2.94(b)}$$

(The last line of equation (2.94(a)) is obvious for real $v(t)$ but see also the more general definition of $R_v(\tau)$ given in equation (2.96(b)).)

5b. The autocorrelation function of a periodic signal and its (two sided) *power* spectral density (represented by a discrete set of impulse functions) are a Fourier transform pair, i.e.:

$$R_p(\tau) \overset{\text{FT}}{\Leftrightarrow} G_p(f) \tag{2.95}$$

(Since $R_p(\tau)$ is periodic and $G_p(f)$ consists of a set of discrete impulse functions, $G_p(f)$ could also be interpreted as a power spectrum derived as the Fourier series of $R_p(\tau)$.) Equation (2.95) also applies to stationary random signals which are discussed in Chapter 3.

EXAMPLE 2.10

What is the autocorrelation function, and decorrelation time, of the rectangular pulse shown in Figure 2.49(a)? From a knowledge of its autocorrelation function find the pulse's energy spectral density.

$$R_v(\tau) = \int_{-\infty}^{\infty} 2\,\Pi(t - 1.5)\; 2\,\Pi(t - 1.5 - \tau)\; dt$$

For $\tau = 0$ (see Figure 2.49(b)):

$$R_v(0) = \int_{1}^{2} 2^2\, dt = 4[\,t\,]_1^2 = 4 \quad (\text{V}^2\ \text{s})$$

For $0 < \tau < 1$ (see Figure 2.49(c)):

$$R_v(\tau) = \int_{1+\tau}^{2} 2^2 \, dt = 4[\, t \,]_{1+\tau}^{2} = 4(1-\tau) \quad (\mathrm{V}^2\ \mathrm{s})$$

For $-1 < \tau < 0$ (see Figure 2.49(d)):

$$R_v(\tau) = \int_{1}^{2+\tau} 2^2 \, dt = 4[\, t \,]_{1}^{2+\tau} = 4(1+\tau) \quad (\mathrm{V}^2\ \mathrm{s})$$

For $|\tau| > 1$ the rectangular pulse and its replica do not overlap; therefore in these regions $R_v(\tau) = 0$. Figure 2.49(e) shows a sketch of $R_v(\tau)$. Notice that the location of this rectangular pulse in time (i.e. at $t = 1.5$) does not affect the location of $R_v(\tau)$ on the time delay axis. Notice also that the symmetry of $R_v(\tau)$ about $\tau = 0$ means that in practice the function need only be found for $\tau > 0$. The decorrelation time, τ_o, of $R_v(\tau)$ is given by:

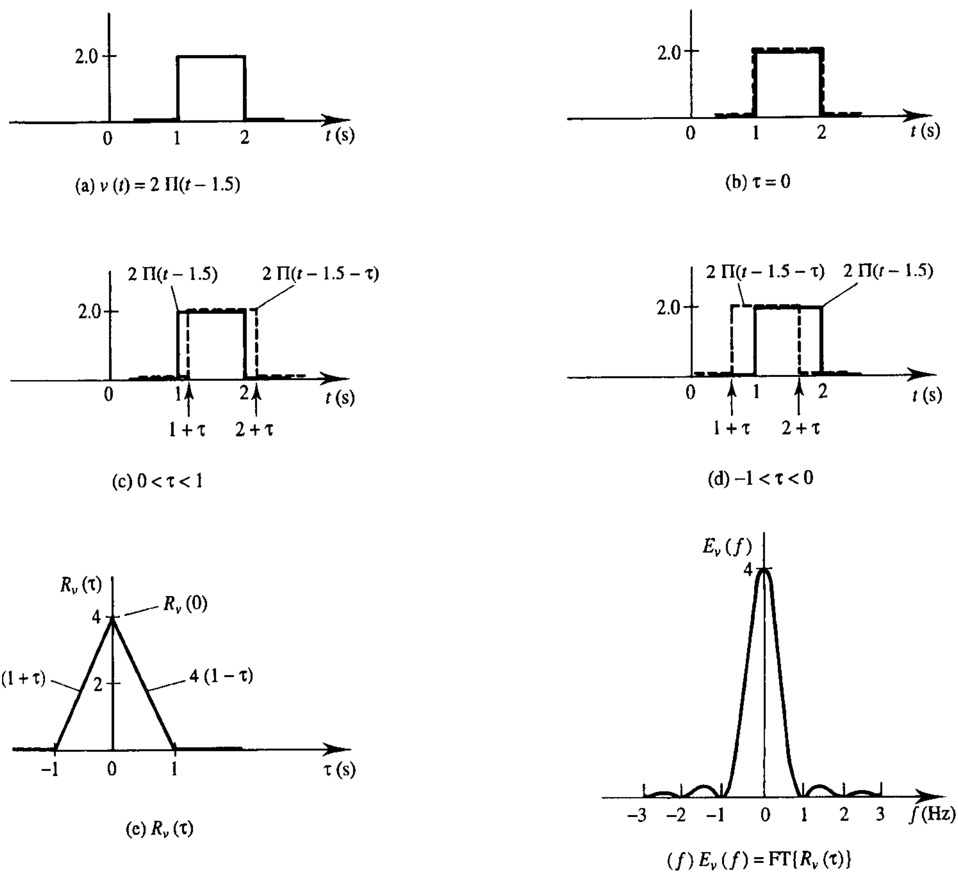


Figure 2.49 *Autocorrelation (e) of a rectangular pulse (a) – (e) and spectral density (f).*

$$\tau_o = 1 \quad (s)$$

(Other definitions for τ_o, e.g. ½ energy, $1/e$ energy, etc., could be adopted.)

The energy spectral density, $E_v(f)$, is given by:

$$E_v(f) = \text{FT}\{R_v(\tau)\} = \text{FT}\{4\,\Lambda(t)\}$$

$$= 4\,\text{sinc}^2(f) \quad (\text{V}^2\ \text{s/Hz}) \quad (\text{Using Table 2.4})$$

$E_v(f)$ is sketched in Figure 2.49(f). Notice that the area under $R_v(\tau)$ is equal to $E_v(0)$ and the area under $E_v(f)$ is equal to $R_v(0)$. This is a good credibility check on the answer to such problems. Also notice that the (first null) bandwidth of the rectangular pulse and the zero crossing definition of decorrelation time are consistent with the rule:

$$B \approx \frac{1}{\tau_o}$$

Note that auto and cross correlation functions are only necessarily real if the functions being correlated are real. If this is not the case then the more general definitions:

$$R_{pq}(\tau) = \lim_{T' \to \infty} \frac{1}{T'} \int_{-T'/2}^{T'/2} p(t)\, q^*(t-\tau)\, dt \tag{2.96(a)}$$

and

$$R_{vw}(\tau) = \int_{-\infty}^{\infty} v(t)\, w^*(t-\tau) dt \tag{2.96(b)}$$

must be adopted.

2.7 Summary

Deterministic signals can be periodic or transient. Periodic waveforms are unchanged when shifted in time by nT seconds where n is any integer and T is the period of the waveform. They have discrete (line) spectra and, being periodic, exist for all time. Transient signals are aperiodic and have continuous spectra. They are essentially localised in time (whether or not they are strictly time limited).

All periodic signals of engineering interest can be expressed as a sum of harmonically related sinusoids. The amplitude spectrum of a periodic signal has units of volts, and the phase spectrum has units of radians or degrees.

Alternatively the amplitudes and phases of a set of harmonically related, counter rotating, conjugate cisoids can be plotted against frequency. This leads naturally to two sided amplitude and phase spectra. For purely real signals the two sided amplitude spectrum has even symmetry about 0 Hz and the phase spectrum has odd symmetry about 0 Hz. If the power associated with each sinusoid in a Fourier series is plotted against frequency the result is a power spectrum with units of V^2 or W. Two sided power spectra can be defined by associating half the total power in each line with a positive frequency and half with a negative frequency. The total power in a waveform is the sum of the powers in each spectral line. This is Parseval's theorem.

Bandwidth refers to the width of the frequency band in a signal's spectrum which contains significant power (or, in the case of transient signals, energy). Many definitions of bandwidth are possible, the most appropriate depending on the application or context. In the absence of a contrary definition, however, the half-power bandwidth is usually assumed. Signals with rapid rates of change have large bandwidth and those with slow rates of change small bandwidth.

The voltage spectrum of a transient signal is continuous and is given by the Fourier transform of the signal. Since the units of such a spectrum are V/Hz it is normally referred to as a voltage spectral density. A complex voltage spectral density can be expressed as an amplitude spectrum and a phase spectrum. The square of the amplitude spectrum has units of V^2 s/Hz and is called an energy spectral density. The total energy in a transient signal is the integral over all frequencies of the energy spectral density.

Fourier transform pairs are uniquely related (i.e. for each time domain signal there is only one, complex, spectrum) and have been extensively tabulated. Theorems allowing the manipulation of existing transform pairs and the calculation of new ones extend the usefulness of such tables. The convolution theorem is especially useful. It specifies the operation in one domain (convolution) which is precisely equivalent to multiplication in the other domain.

Basis functions other than sinusoids and cisoids can be used to expand signals and waveforms. Such generalised expansions are especially useful when the set of basis functions are orthogonal or orthonormal. Signals and waveforms can be interpreted as multidimensional vectors. In this context the concept of orthogonal functions is related to the concept of perpendicular vectors. The orthogonal property of a set of basis functions allows the optimum coefficients of the functions to be calculated independently. Optimum in this context means a minimum error energy approximation.

Correlation is the equivalent operation for signals to the scalar product for vectors and is a measure of signal similarity. The cross correlation function gives the correlation of two functions for all possible time shifts between them. It can be applied, with appropriate differences in its definition, to both transient and periodic functions. The energy and power spectral densities of transient and periodic signals respectively are the Fourier transforms of their autocorrelation functions. Chapter 3 extends correlation concepts to noise and other random signals. In Chapter 8 correlation is identified as an optimum signal processing technique, often employed in digital communications receivers.

2.8 Problems

2.1. Find the DC component and the first two non-zero harmonic terms in the Fourier series of the following periodic waveforms: (a) square wave with period 20 ms and magnitude +2 V from −5 ms to +5 ms and −2 V from +5 ms to +15 ms; (b) sawtooth waveform with a 2 s period and $y = t$ for

$-1 \le t < 1$; and (c) triangular wave with 0.2 s period and $y = 1/3(1 - 10|t|)$ for $-0.1 \le t < 0.1$.

2.2. Find the proportion of the total power contained in the DC and first two harmonics of the waveforms shown in Table 2.2, assuming a 25% duty cycle for the third waveform.

2.3. Use Table 2.2 to find the Fourier series coefficients up to the third harmonic of the waveform with period 2 s, one period of which is formed by connecting the following points with straight lines: (t, y) = (0,0), (0,0.5), (1,1), (1,0.5), (2,0). (Hint: decompose the waveform into a sum of waveforms which you recognise.)

2.4. The spectrum of a square wave, amplitude ± 1.0 V and period 1.0 ms is bandlimited by an ideal filter to 4.0 kHz such that frequencies below 4.0 kHz are passed (undistorted) and frequencies above 4.0 kHz are stopped. What is the normalised power (in V^2), and what is the maximum rate of change (in V/s), of the waveform at the output of the filter? [0.90 V^2, 16×10^3 V/s]

2.5. How fast must the bandlimited waveform in Problem 2.4 be sampled if (in the absence of noise) it is to be reconstructed from the samples without error? [8.0 kHz]

2.6. Find (without using Table 2.4) the Fourier transforms of the following functions: (a) $\Pi((t - T)/\tau)$; (b) $\Lambda(t/2)$; (c) $3e^{-5|t|}$; (d) $[(e^{-at} - e^{-bt})/(b - a)]\, u(t)$. [Hint: for (c) recall the standard integral $- \int e^{ax} \cos bx \, dx = \dfrac{e^{ax}(a \cos bx + b \sin bx)}{a^2 + b^2}$].

2.7. Find (using Tables 2.4 and 2.5) the amplitude and phase spectra of the following transient signals: (a) a triangular pulse $3(1 - |t - 1|)\Pi((t - 1)/2)$; (b) a 'split phase' rectangular pulse having amplitude $y = 2$ V from $t = 0$ to $t = 1$ and $y = -2$ V from $t = 1$ to $t = 2$ and $y = 0$ elsewhere; (c) a truncated cosine wave $\cos(2\pi 20t)\Pi(t/0.2)$; (d) an exponentially decaying sinusoid $u(t)e^{-5t}\sin(2\pi 20t)$.

2.8. Sketch the following, purely real, frequency spectra and find the time domain signals to which they correspond: (a) $0.1\,\text{sinc}(3f)$; (b) e^{-f^2}; (c) $\Lambda(f/2) + \Pi(f/4)$; and (d) $\Lambda(f - 10) + \Lambda(f + 10)$.

2.9. Convolve the following pairs of signals: (a) $\Pi(t/T_2)/T_2$ with $\Pi(t/T_1)/T_1$, $(T_2 > T_1)$; (b) $u(t)\exp(-3t)$ with $u(t - 1)$; (c) $\sin(\pi t)\Pi((t - 1)/2)$ with $2\Pi((t - 2)/2)$; and (d) $\delta(t) - 2\delta(t - 1) + \delta(t - 2)$ with $\Pi(t - 0.5) + 2\Pi(t - 1.5)$.

2.10. Find and sketch the energy spectral densities of the following signals: (a) $10\Pi((t - 0.05)/0.1)$; (b) $6\,e^{-6|t|}$; (c) $\text{sinc}(100t)$; (d) $-\text{sinc}(100t)$. What is the energy contained in signals (a) and (c), and how much energy is contained in signals (b) and (d) below a frequency of 6.0 Hz? [10 V^2s, 0.01 V^2s, 5.99 V^2s, 1.2×10^{-3} V^2s}

2.11. Demonstrate the orthogonality, or otherwise, of the function set: $(1/\sqrt{T})\,\Pi((t - T/2)/T)$; $(\sqrt{2/T})\cos((\pi/T)\,t)\,\Pi((t - T/2)/T)$; $2\Lambda((t - T/2)/(T/2)) - \Pi((t - T/2)T)$. Do these functions represent an orthonormal set?

2.12. Find the cross-correlation function of the sinewave, $f(t) = \sin(2\pi 50t)$, with a half wave rectified version of itself, $g(t)$.

2.13. Find, and sketch, the autocorrelation function of the 'split phase' rectangular pulse, where $x(t) = -V_0$ for $-T/2 < t < 0$ and $+V_0$ for $0 < t < T/2$.

2.14. What is the autocorrelation of $v(t) = u(t)e^{-t}$? Find the energy spectral density of this signal and the proportion of its energy contained in frequencies above 2.0 Hz. [5.1%]